# Generic Properties of Koopman Eigenfunctions for Stable Fixed Points and Periodic Orbits ⋆

**Matthew D. Kvalheim** * **Shai Revzen** **

* *Department of Electrical and Systems Engineering, University of Pennsylvania, Philadelphia, PA 19104 USA (e-mail: kvalheim@seas.upenn.edu).*
** *Department of Electrical Engineering and Computer Science, Ecology and Evolutionary Biology Department, Robotics Institute, University of Michigan, Ann Arbor, MI 48109 (e-mail: shrevzen@umich.edu)*

**Abstract:** Our recent work established existence and uniqueness results for $C^{k,\alpha}_{\mathrm{loc}}$ globally defined linearizing semiconjugacies for $C^1$ flows having a globally attracting hyperbolic fixed point or periodic orbit (Kvalheim and Revzen, 2019). Applications include (i) various improvements—such as uniqueness statements—for the Sternberg linearization and Floquet normal form theorems, and (ii) results concerning the existence, uniqueness, classification, and convergence of various quantities appearing in the "applied Koopmanism" literature (such as principal eigenfunctions, isostables, and Laplace averages).

In this work we give an exposition of some of these results, with an emphasis on the Koopman applications, and then consider their broadness of applicability. In particular we show that, for "almost all" $C^\infty$ flows having a globally attracting hyperbolic fixed point or periodic orbit, the $C^\infty$ Koopman eigenfunctions can be completely classified in a simple way generalizing one known for analytic systems. In particular, for such systems every $C^\infty$ eigenfunction is uniquely determined by its eigenvalue modulo scalar multiplication.

*Keywords:* Koopman operator, eigenfunctions, generic properties, isostables, periodic orbits
*AMS subject classification:* 37C10, 37C15, 37C20

## 1. INTRODUCTION

In this paper, we discuss Koopman eigenfunctions of $C^1$ flows $\Phi\colon Q \times \mathbb{R} \to Q$ having a globally attracting hyperbolic fixed point or periodic orbit. Here $Q$ is a smooth manifold; a common example is that of $t \mapsto \Phi^t(x_0)$ being the solution to the initial value problem

$$\frac{d}{dt}x(t) = f(x(t)), \qquad x(0) = x_0$$

determined by a complete $C^1$ vector field $f$ on $Q$.

In our previous work (Kvalheim and Revzen, 2019) we obtained existence and uniqueness results regarding globally defined $C^k$ linearizing semiconjugacies $\psi\colon Q \to \mathbb{C}^m$ which, by definition, make the diagram

$$\begin{tikzcd} Q \arrow[r, "\Phi^t"] \arrow[d, "\psi"] & Q \arrow[d, "\psi"] \\ \mathbb{C}^m \arrow[r, "e^{tA}"] & \mathbb{C}^m \end{tikzcd} \tag{1}$$

commute for some $A \in \mathbb{C}^{m\times m}$ and all $t \in \mathbb{R}$. Several applications of these results—including uniqueness of Sternberg linearizations and Floquet normal forms—were discussed by Kvalheim and Revzen (2019), but in this work we focus on the applications to Koopman eigenfunctions, the special case of linearizing semiconjugacies in (1) with $m = 1$.

The eigenfunction results of Kvalheim and Revzen (2019) are most relevant to the "applied Koopmanism" literature largely initiated by Dellnitz and Junge (1999); Mezić and Banaszuk (2004); Mezić (2005) nearly a century after Koopman's seminal work (Koopman, 1931). [1] More specifically, these results yield precise conditions under which quantities such as *principal eigenfunctions*, *isostables* for point attractors (Mohr and Mezić, 2016; Mauroy et al., 2013), and *isostable coordinates* for periodic orbit attractors (Wilson and Moehlis, 2016; Shirasaka et al., 2017; Wilson and Ermentrout, 2018; Monga et al., 2019) exist and are unique. Such quantities have been the targets of numerical algorithms, and their existence and uniqueness is relevant to the well-posedness of such algorithms.

The contribution of the present work is to show that the existence and uniqueness results for $C^\infty$ eigenfunctions in Kvalheim and Revzen (2019) hold, in some sense, for "most" $C^\infty$ vector fields having a globally asymptotically

⋆ The majority of this work was performed while Kvalheim was a postoctoral researcher at the University of Michigan. Both authors were supported by ARO grants W911NF-14-1-0573 and W911NF-17-1-0306 to Revzen.

[1] See the references in Kvalheim and Revzen (2019) for many additional examples of this literature omitted here due to space constraints.

stable equilibrium or periodic orbit. More precisely, we show that the set of such vector fields for which the $C^\infty$ results hold is open in the $C^1$ compact-open topology and dense in the $C^\infty$ Whitney topology. Moreover, we show that the set of linearizations of such vector fields has full Lebesgue measure.

The remainder of the paper is organized as follows. In §2 we first give preliminary definitions and discuss the results of Kvalheim and Revzen (2019) relevant for Koopman eigenfunctions. In §3 we prove most of the genericity results. Finally, Appendix A contains the proof of the measure-theoretic result mentioned above.

Lastly, we mention that the results of Kvalheim and Revzen (2019) are stated in terms of $C^{k,\alpha}_{\text{loc}}$ functions, but in this work we simplify matters by stating less general results in terms of $C^k$ functions only.

## 2. PREVIOUS RESULTS

In §2.3 and 2.4 we present results from Kvalheim and Revzen (2019, Sec. 3). Before that, we define Koopman eigenfunctions and principal eigenfunctions in §2.1. §2.2 contains preliminary definitions needed to state the results in §2.3 and 2.4.

We use the following notation in the sequel. Given a differentiable map $F\colon M \to N$ between smooth manifolds, in the remainder of this paper we use the notation $\mathsf{D}_x F$ for the derivative of $F$ at the point $x \in M$. (Recall that the derivative $\mathsf{D}_x F\colon \mathsf{T}_x M \to \mathsf{T}_{F(x)} N$ is a linear map between tangent spaces (Lee, 2013), which can be identified with the Jacobian of $F$ evaluated at $x$ in local coordinates.) In particular, given a flow $\Phi\colon Q \times \mathbb{R} \to Q$ and fixed $t \in \mathbb{R}$, we write $\mathsf{D}_x \Phi^t\colon \mathsf{T}_x Q \to \mathsf{T}_{\Phi^t(x)} Q$ for the derivative of the time-$t$ map $\Phi^t\colon Q \to Q$ at the point $x \in Q$.

### *2.1 Koopman eigenfunctions*

Given a $C^1$ flow $\Phi\colon Q \times \mathbb{R} \to \mathbb{R}$, where $Q$ is a smooth manifold, we say that $\psi\colon Q \to \mathbb{C}$ is a *Koopman eigenfunction* with *eigenvalue* $\mu \in \mathbb{C}$ if $\psi$ is not identically zero and satisfies

$$\forall t \in \mathbb{R}\colon \psi \circ \Phi^t = e^{\mu t}\psi. \tag{2}$$

The following are intrinsic definitions of principal eigenfunctions and principal algebras which extend the definitions for linear systems given in Mohr and Mezić (2016, Def. 2.2–2.3). The condition $\psi|_M \equiv 0$ was motivated in part by the definition of a certain space $\mathcal{F}_{A_c}$ of functions in Mauroy and Mezić (2016, p. 3358). Suppose that $\Phi$ has a distinguished, closed, invariant subset $M \subset Q$. We say that an eigenfunction $\psi \in C^1(Q)$ is a *principal* eigenfunction if

$$\psi|_M \equiv 0 \qquad \text{and} \qquad \forall x \in M\colon \mathsf{D}_x\psi \neq 0. \tag{3}$$

We define the $C^k$ principal algebra $\mathcal{A}^k_\Phi$ to be the complex subalgebra of $C^k(Q, \mathbb{C})$ generated by all $C^k$ principal eigenfunctions.

*Remark 1.* In the case that $\Phi|_{M\times\mathbb{R}}$ is minimal (has no proper, closed, nonempty invariant subsets)—(3) can be replaced by the weaker condition

$$\exists x \in M\colon \psi(x) = 0 \qquad \text{and} \qquad \exists y \in M\colon \mathsf{D}_y\psi \neq 0. \tag{4}$$

This will be the case in the sequel, in which we consider only the cases that $M$ is either a fixed point or periodic orbit.

### *2.2 Nonresonance and spectral spread*

The following two definitions are essentially asymmetric versions of some appearing in Sternberg (1957); Sell (1985). When discussing eigenvalues and eigenvectors of a linear map or matrix in this work, we always mean eigenvalues and eigenvectors of its complexification, although we do not always make this explicit.

*Definition 2.* ($(X, Y)$ $k$-nonresonance). Let $X \in \mathbb{C}^{d\times d}$ and $Y \in \mathbb{C}^{n\times n}$ be matrices with eigenvalues $\mu_1, \ldots, \mu_d$ and $\lambda_1, \ldots, \lambda_n$, respectively, repeated with multiplicities. For any $k \in \mathbb{N}_{\geq 1}$, we say that $(X, Y)$ is *$k$-nonresonant* if, for any $i \in \{1, \ldots, d\}$ and any $m = (m_1, \ldots, m_n) \in \mathbb{N}^n_{\geq 0}$ satisfying $2 \leq m_1 + \cdots + m_n \leq k$,

$$\mu_i \neq \lambda_1^{m_1} \cdots \lambda_n^{m_n}. \tag{5}$$

(Note this condition vacuously holds if $k = 1$.) We say $(X, Y)$ is *$\infty$-nonresonant* if $(X, Y)$ is $k$-nonresonant for every $k \in \mathbb{N}_{\geq 1}$.

For the definition below, recall that the spectral radius $\rho(X)$ of a matrix is defined to be the largest modulus (absolute value) of the eigenvalues of (the complexification of) $X$.

*Definition 3.* ($(X, Y)$ spectral spread). Let $X \in \mathsf{GL}(m, \mathbb{C})$ and $Y \in \mathsf{GL}(n, \mathbb{C})$ be invertible matrices with the spectral radius $\rho(Y)$ satisfying $\rho(Y) < 1$. We define the spectral spread $\nu(X, Y)$ to be

$$\nu(X, Y) := \max_{\substack{\mu \in \text{spec}(X) \\ \lambda \in \text{spec}(Y)}} \frac{\ln(|\mu|)}{\ln(|\lambda|)}. \tag{6}$$

### *2.3 Principal eigenfunctions for fixed points and periodic orbits*

Differentiating (2) and using the chain rule immediately yields Propositions 4 and 5, which have appeared in the literature (see e.g. the proof of (Mauroy and Mezić, 2016, Prop. 2)). In these results, $d$ denotes the differential of a function and $\mathsf{T}^*_{x_0} Q$ denotes the cotangent space to $x_0$; $d\psi(x_0)$ corresponds to $\mathsf{D}_{x_0}\psi$ after making the canonical identification $\mathbb{C} \cong \mathsf{T}_{\psi(x_0)}\mathbb{C}$.

*Proposition 4.* Let $x_0$ be a fixed point of the $C^1$ flow $\Phi\colon Q \times \mathbb{R} \to Q$. If $\psi$ is a principal Koopman eigenfunction for $\Phi$ satisfying with eigenvalue $\mu \in \mathbb{C}$, then for any $t \in \mathbb{R}$, it follows that $d\psi(x_0) \in (T^*_{x_0} Q)_{\mathbb{C}}$ is an eigenvector of the (complexified) adjoint $(\mathsf{D}_{x_0}\Phi^t)^*$ with eigenvalue $e^{\mu t}$.

*Proposition 5.* Let $\Gamma$ be the image of a $\tau$-periodic orbit of the $C^1$ flow $\Phi\colon Q \times \mathbb{R} \to Q$. If $\psi$ is a principal Koopman eigenfunction for $\Phi$ with eigenvalue $\mu \in \mathbb{C}$, then for any $x_0 \in \Gamma$, it follows that $d\psi(x_0) \in (T^*_{x_0} Q)_{\mathbb{C}}$ is an eigenvector of the (complexified) adjoint $(\mathsf{D}_{x_0}\Phi^\tau)^*$ with eigenvalue $e^{\mu\tau}$; in particular, $e^{\mu\tau}$ is a Floquet multiplier for $\Gamma$.

The following result is Kvalheim and Revzen (2019, Prop. 6) and uses the following notation. Given $i \geq 0$, we let $\mathsf{D}^i_x F$ denote the $i$-th derivative of $F$, which can be identified with an $i$-multilinear map $\mathsf{D}^i_x F\colon (\mathsf{T}_x M)^i \to \mathsf{T}_{F(x)} N$ from $(\mathsf{T}_x M)^i$ to $\mathsf{T}_x N$. In local coordinates, $\mathsf{D}^i F$ is represented by the $(1 + i)$-dimensional array of $i$-th partial derivatives of $F$ evaluated at $x$. The result statement also mentions the $C^k$ compact-open (weak) topology (Hirsch, 1994, Ch. 2) on functions, which is the topology of uniform convergence of a function and its first $k$ derivatives on compact sets.

*Proposition 6.* Let $\Phi\colon Q \times \mathbb{R} \to Q$ be a $C^1$ flow having a globally attracting hyperbolic fixed point $x_0 \in Q$. Fix $k \in \mathbb{N}_{\geq 1} \cup \{+\infty\}$, fix $\mu \in \mathbb{C}$, and let $\psi_1 \in C^k(Q, \mathbb{C})$ be any Koopman eigenfunction with eigenvalue $\mu \in \mathbb{C}$.

***Uniqueness of Koopman eigenvalues and principal eigenfunctions.*** Assume that $\nu(e^\mu, \mathsf{D}_{x_0}\Phi^1) \leq k$.

(1) Then there exists $m = (m_1, \ldots, m_n) \in \mathbb{N}^n_{\geq 0}$ such that
$$e^\mu = e^{m \cdot \lambda},$$
where $e^{\lambda_1}, \ldots, e^{\lambda_n}$ are the eigenvalues of $\mathsf{D}_{x_0}\Phi^1$ repeated with multiplicities and $\lambda := (\lambda_1, \ldots, \lambda_n)$.

(2) Additionally assume that $\psi_1$ is a principal eigenfunction so that $e^\mu \in \operatorname{spec}(\mathsf{D}_{x_0}\Phi^1)$, and assume that $(e^\mu, \mathsf{D}_{x_0}\Phi^1)$ is $k$-nonresonant. Then $\psi_1$ is uniquely determined by $d\psi_1(x_0)$, and if $\mu$ and $d\psi_1(x_0)$ are real, then $\psi\colon Q \to \mathbb{R} \subset \mathbb{C}$ is real. In particular, if $e^\mu$ is an algebraically simple eigenvalue of (the complexification of) $\mathsf{D}_{x_0}\Phi^1$ and if $\psi_2$ is any other principal eigenfunction with eigenvalue $\mu$, then there exists $c \in \mathbb{C} \setminus \{0\}$ such that
$$\psi_1 = c\psi_2.$$

***Existence of principal eigenfunctions.*** Assume that $\Phi \in C^k$, that $e^\mu \in \operatorname{spec}(\mathsf{D}_{x_0}\Phi^1)$, that $(e^\mu, \mathsf{D}_{x_0}\Phi^1)$ is $k$-nonresonant, and that $\nu(e^\mu, \mathsf{D}_{x_0}\Phi^1) < k$. Let $w \in (\mathsf{T}^*_{x_0}Q)_\mathbb{C}$ be any eigenvector of the (complexified) adjoint $(\mathsf{D}_{x_0}\Phi^1)^*$ with eigenvalue $e^\mu$.

(1) Then there exists a unique principal eigenfunction $\psi \in C^k(Q, \mathbb{C})$ with eigenvalue $\mu$ and satisfying $d\psi(x_0) = w$.

(2) In fact, if $P$ is any "approximate eigenfunction" satisfying $\mathsf{D}_{x_0}P = w$ and
$$P \circ \Phi^1 = e^\mu P + R \tag{7}$$
with $\mathsf{D}^i_{x_0}R = 0$ for all $0 \leq i < k$, then
$$\psi = \lim_{t \to \infty} e^{-\mu t} P \circ \Phi^t, \tag{8}$$
in the $C^k$ compact-open topology.

*Remark 7.* (the $C^\infty$ case). In the case that $k = \infty$, the hypotheses $\nu(e^A, \mathsf{D}_{x_0}\Phi^1) < k$ and $\nu(e^A, \mathsf{D}_{x_0}\Phi^1) \leq k$ become $\nu(e^A, \mathsf{D}_{x_0}\Phi^1) < \infty$ and $\nu(e^A, \mathsf{D}_{x_0}\Phi^1) \leq \infty$, which are automatically satisfied since $\nu(e^A, \mathsf{D}_{x_0}\Phi^1)$ is always finite. Hence for the case $k = \infty$, no assumption is needed on the spectral spread in Proposition 6 (and Proposition 10 below); we need only assume that $(e^A, \mathsf{D}_{x_0}\Phi^1)$ is $\infty$-nonresonant.

*Remark 8.* (Laplace averages). Given $P\colon Q \to \mathbb{C}$, in the Koopman literature the *Laplace average*
$$\psi := \lim_{T \to \infty} \frac{1}{T} \int_0^T e^{-\mu t} P \circ \Phi^t \, dt$$
is used to produce a Koopman eigenfunction with eigenvalue $\mu$ as long as the limit exists Mauroy et al. (2013); Mohr and Mezić (2014). Since convergence of the limit (8) clearly implies convergence of the Laplace average to the same limiting function, Proposition 6 yields sufficient conditions under which the Laplace average of $P$ exists and is equal to a $C^k$ principal eigenfunction satisfying $\mathsf{D}_{x_0}P = w$.

*Remark 9.* For a detailed discussion on how Propositions 6 and 10 below relate to the literature on isostables and isostable coordinates (Mauroy et al., 2013; Wilson and Moehlis, 2016; Shirasaka et al., 2017), including the convergence of certain limits in this literature, see Kvalheim and Revzen (2019, Remark 10). For a discussion relating Proposition 6 to the work of Mohr and Mezić (2016) on principal eigenfunctions, see Kvalheim and Revzen (2019, Rem 11).

The following result is Kvalheim and Revzen (2019, Prop. 7). For an illustration of the conditions involving $\nu(e^{\mu\tau}, \mathsf{D}_{x_0}\Phi^\tau|_{E^s_{x_0}})$ below, see Figure 1.

*Proposition 10.* Fix $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$ and let $\Phi\colon Q \times \mathbb{R} \to Q$ be a $C^k$ flow having a globally attracting hyperbolic $\tau$-periodic orbit with image $\Gamma \subset Q$. Fix $x_0 \in \Gamma$ and let $E^s_{x_0}$ denote the unique $\mathsf{D}_{x_0}\Phi^\tau$-invariant subspace complementary to $\mathsf{T}_{x_0}\Gamma$. Let $\psi_1 \in C^k(Q, \mathbb{C})$ be any Koopman eigenfunction with eigenvalue $\mu \in \mathbb{C}$.

***Uniqueness of Koopman eigenvalues.*** Assume that $\nu(e^{\mu\tau}, \mathsf{D}_{x_0}\Phi^\tau|_{E^s_{x_0}}) \leq k$.

(1) Then there exists $m = (m_1, \ldots, m_n) \in \mathbb{N}^n_{\geq 0}$ such that
$$\mu \in m \cdot \lambda + \frac{2\pi j}{\tau}\mathbb{Z},$$
where $e^{\lambda_1\tau}, \ldots, e^{\lambda_n\tau}$ are the eigenvalues of $\mathsf{D}_{x_0}\Phi^\tau|_{E^s_{x_0}}$ repeated with multiplicities and $\lambda := (\lambda_1, \ldots, \lambda_n)$.

(2) Additionally assume that $\psi_1$ is a principal eigenfunction so that $e^{\mu\tau} \in \operatorname{spec}(\mathsf{D}_{x_0}\Phi^\tau|_{E^s_{x_0}})$, and assume that $(e^{\mu\tau}, \mathsf{D}_{x_0}\Phi^\tau|_{E^s_{x_0}})$ is $k$-nonresonant. Then $\psi_1$ is uniquely determined by $d\psi_1(x_0)$, and if $\mu$ and $d\psi_1(x_0)$ are real, then $\psi\colon Q \to \mathbb{R} \subset \mathbb{C}$ is real. In particular, if $e^\mu$ is an algebraically simple eigenvalue of (the complexification of) $\mathsf{D}_{x_0}\Phi^1$ and if $\psi_2$ is any other principal eigenfunction with eigenvalue $\mu$, then there exists $c \in \mathbb{C} \setminus \{0\}$ such that
$$\psi_1 = c\psi_2.$$

***Existence of principal eigenfunctions.*** Assume that $e^{\mu\tau} \in \operatorname{spec}(\mathsf{D}_{x_0}\Phi^\tau|_{E^s_{x_0}})$, that $(e^{\mu\tau}, \mathsf{D}_{x_0}\Phi^\tau|_{E^s_{x_0}})$ is $k$-nonresonant, and that $\nu(e^{\mu\tau}, \mathsf{D}_{x_0}\Phi^\tau|_{E^s_{x_0}}) < k$. Let $w \in (E^s_{x_0})^*_\mathbb{C}$ be any eigenvector of the (complexified) adjoint $(\mathsf{D}_{x_0}\Phi^\tau|_{E^s_{x_0}})^*$ with eigenvalue $e^{\mu\tau}$. Then there exists a unique principal eigenfunction $\psi_1 \in C^k(Q, \mathbb{C})$ for $\Phi$ with eigenvalue $\mu$ and satisfying $d\psi_1(x_0)|_{E^s_{x_0}} = w$.

For an example demonstrating that Propositions 6 and 10 are reasonably sharp, we refer the reader to Kvalheim and Revzen (2019, Example 1).

### *2.4 Classification of all $C^\infty$ Koopman eigenfunctions*

To improve the readability of Theorems 11 and 12 below, we introduce the following multi-index notation. We define an $n$-dimensional multi-index to be an $n$-tuple $i = (i_1, \ldots, i_n) \in \mathbb{N}^n_{\geq 0}$ of nonnegative integers, and define its sum to be $|i| := i_1 + \ldots i_n$. For a multi-index $i \in \mathbb{N}^n_{\geq 0}$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we define $z^{[i]} := z_1^{i_1} \cdots z_n^{i_n}$. Given a $\mathbb{C}^n$-valued function $\psi = (\psi_1, \ldots, \psi_n)\colon Q \to \mathbb{C}^n$, we define $\psi^{[i]}\colon Q \to \mathbb{C}$ via $\psi^{[i]}(x) := (\psi(x))^{[i]}$ for all $x \in Q$.

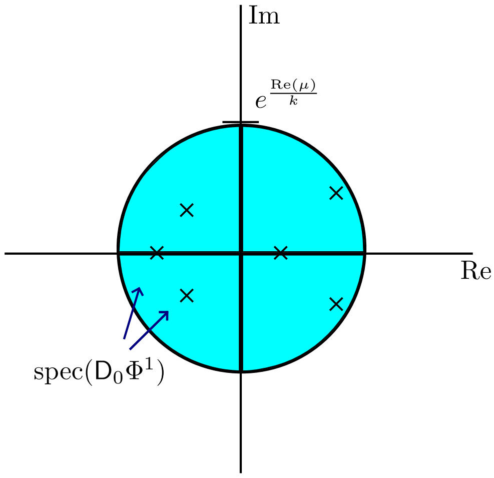


Fig. 1. An illustration of the condition $\nu(e^{\mu}, \mathsf{D}_0\Phi^1) < k$ of Proposition 6. Unwinding Definition 3, it follows that this condition is equivalent to every eigenvalue of $\mathsf{D}_0\Phi^1$ (represented by an "$\times$" above) belonging to the open disk with radius $e^{\frac{\mathrm{Re}(\mu)}{k}}$.

We also define the complex conjugate of $\psi = (\psi_1, \ldots, \psi_n)$ element-wise: $\bar{\psi} := (\bar{\psi}_1, \ldots, \bar{\psi}_n)$.

The following result is Kvalheim and Revzen (2019, Thm 3).

*Theorem 11.* (Classification for a point attractor). Let $\Phi\colon Q \times \mathbb{R} \to Q$ be a $C^\infty$ flow having a globally attracting hyperbolic fixed point $x_0 \in Q$. Assume that $\mathsf{D}_{x_0}\Phi^1$ is semisimple and that $(\mathsf{D}_{x_0}\Phi^1, \mathsf{D}_{x_0}\Phi^1)$ is $\infty$-nonresonant.

Letting $n = \dim(Q)$, it follows that there exists an $n$-tuple

$$\psi = (\psi_1, \ldots, \psi_n)$$

of $C^\infty$ principal eigenfunctions such that every $C^\infty$ Koopman eigenfunction $\varphi$ is a (finite) sum of scalar multiples of products of the $\psi_i$ and their complex conjugates $\bar{\psi}_i$:

$$\varphi = \sum_{|i|+|\ell|\leq k} c_{i,\ell}\psi^{[i]}\bar{\psi}^{[\ell]} \tag{9}$$

for some $k \in \mathbb{N}_{\geq 1}$ and some coefficients $c_{i,\ell} \in \mathbb{C}$.

For a globally attracting hyperbolic $\tau$-periodic orbit of a $C^\infty$ flow with image $\Gamma$, let $W^s_{x_0}$ be the global strong stable manifold (isochron) through a point $x_0 \in \Gamma$.[2] As discussed in Kvalheim and Revzen (2019), there is a unique continuous eigenfunction with eigenvalue $\mu = \frac{2\pi}{\tau}$ satisfying $\psi_\theta|_{W^s_{x_0}} \equiv 1$, and this eigenfunction is in fact $C^\infty$. We use this notation in the theorem below, where $x_0$ is as in the theorem statement.

The following result is Kvalheim and Revzen (2019, Thm 4).

*Theorem 12.* (Classification for a limit cycle attractor). Let $\Phi\colon Q \times \mathbb{R} \to Q$ be a $C^\infty$ flow having a globally attracting hyperbolic $\tau$-periodic orbit with image $\Gamma \subset Q$. Fix $x_0 \in \Gamma$ and denote by $E^s_{x_0}$ the unique $\tau$-invariant subspace complementary to $\mathsf{T}_{x_0}\Gamma$. Assume that $\mathsf{D}_{x_0}\Phi^\tau|_{E^s_{x_0}}$ is semisimple and that $(\mathsf{D}_{x_0}\Phi^\tau|_{E^s_{x_0}}, \mathsf{D}_{x_0}\Phi^\tau|_{E^s_{x_0}})$ is $\infty$-nonresonant.

Letting $n + 1 = \dim(Q)$, it follows that there exists an $n$-tuple

$$\psi = (\psi_1, \ldots, \psi_n)$$

of $C^\infty$ principal eigenfunctions such that every $C^\infty$ Koopman eigenfunction $\varphi$ is a (finite) sum of scalar multiples of products of integer powers of $\psi_\theta$ with products of the $\psi_i$ and their complex conjugates $\bar{\psi}_i$:

$$\varphi = \sum_{|\ell|+|m|\leq k} c_{\ell,m}\psi^{[\ell]}\bar{\psi}^{[m]}\psi_\theta^{j_{\ell,m}} \tag{10}$$

for some $k \in \mathbb{N}_{\geq 1}$, some coefficients $c_{\ell,m} \in \mathbb{C}$, and $j_{\ell,m} \in \mathbb{Z}$.

[2] See, e.g., Fenichel (1974, 1977); Hirsch et al. (1977); de la Llave and Wayne (1995); Kvalheim and Revzen (2019).

## 3. GENERICITY OF THE $C^\infty$ RESULTS

Let $\mathcal{H}_n \subset \mathsf{GL}(n, \mathbb{R})$ denote the set of Hurwitz matrices. Let $\mathcal{N}_n \subset \mathsf{GL}(n, \mathbb{R})$ denote the set of real invertible matrices $A$ *with distinct eigenvalues* such that $(A, A)$ is $\infty$-nonresonant.

*Proposition 13.* The Lebesgue measure of $\mathcal{H}_n \setminus \mathcal{N}_n$ is zero, and $\mathcal{N}_n \cap \mathcal{H}_n$ is dense in $\mathcal{H}_n$. Furthermore, $\mathcal{N}_n \cap \mathcal{H}_n$ is open in $\mathcal{H}_n$.

**Proof.** In Appendix A we show that $\mathsf{GL}(n, \mathbb{R}) \setminus \mathcal{N}_n$ has measure zero (Corollary 20), and $\mathcal{H}_n$ is an open subset of $\mathsf{GL}(n, \mathbb{R})$, so it follows that $\mathcal{H}_n \setminus \mathcal{N}_n$ has measure zero. Density of $\mathcal{N}_n \cap \mathcal{H}_n$ in $\mathcal{H}_n$ follows (Lee, 2013, Prop. 6.8).

It remains to prove openness. Fix any $A \in \mathcal{N}_n$. Since $\nu(A, A)$ is always finite, we have $\nu(A, A) < k + 1$ for some $k \in \mathbb{N}$. Now it follows from Definitions 2 and 3 that $\infty$-nonresonance of $(B, B)$ is implied by (i) $k$-nonresonance of $(B, B)$ and (ii) $\nu(B, B) < k + 1$. Since the eigenvalues of a matrix depend continuously on the matrix (Palis and De Melo, 1982, p. 53), the set of matrices satisfying each of these latter conditions is open. Hence $A$ has a neighborhood in $\mathcal{H}_n$ contained in $\mathcal{N}_n$ as desired. This completes the proof.

Let $\mathfrak{X}^\infty_{\mathrm{fix}}(Q)$ (resp. $\mathfrak{X}^\infty_{\mathrm{per}}(Q)$) be the set of $C^\infty$ vector fields $f$ whose flows possess a globally asymptotically stable fixed point $x_f$ (resp. periodic orbit $\Gamma_f$). We use the notation $\Phi_f$ for the flow of such a vector field $f$. We refer the reader to Hirsch (1994, Ch. 2) for the definitions of the $C^k$ Whitney (strong) and compact-open (weak) topologies used in the following results.

*Lemma 14.* The subset of vector fields $f \in \mathfrak{X}^\infty_{\mathrm{fix}}(Q)$ for which $(\mathsf{D}_{x_f}\Phi_f, \mathsf{D}_{x_f}\Phi_f)$ is $\infty$-nonresonant is open in $\mathfrak{X}^\infty_{\mathrm{fix}}(Q)$ with respect to the $C^1$ compact-open topology, and dense in $\mathfrak{X}^\infty_{\mathrm{fix}}(Q)$ with respect to the $C^\infty$ Whitney topology.

**Proof.** The theorem holds vacuously if $\mathfrak{X}^\infty_{\mathrm{fix}}(Q) = \varnothing$; if $\mathfrak{X}^\infty_{\mathrm{fix}}(Q) \neq \varnothing$ then $Q$ is diffeomorphic to $\mathbb{R}^n$ (Wilson, 1967), so we may henceforth assume that $Q = \mathbb{R}^n$.

*Density.* Let $f \in \mathfrak{X}^\infty_{\mathrm{fix}}(\mathbb{R}^n)$ be arbitrary and let $U \subset Q$ be a precompact neighborhood of $x_f$. Let $\varphi\colon \mathbb{R}^n \to [0, \infty)$ be a $C^\infty$ function equal to 1 on a neighborhood of $x_f$ and having support contained in $U$. The density portion of Proposition 13 yields a sequence $(e^{A_n})_{n\in\mathbb{N}}$ of nonresonant matrices converging to $e^{\mathsf{D}_{x_f} f} = \mathsf{D}_{x_f}\Phi_f^1$ in $\mathcal{H}_n$. We define a sequence $(g_n)_{n\in\mathbb{N}}$ of $C^\infty$ vector fields by

$$g_n := f + \varphi \cdot (A_n - \mathsf{D}_{x_f} f)$$

converges to $f$ in the $C^\infty$ Whitney topology. All derivatives of the $g_n$ converge uniformly to those of $f$ on $U$, and $g_n$ is equal to $f$ on $\mathbb{R}^n \setminus U$, so $g_n$ converges to $f$ in the $C^\infty$ Whitney toplogy. It remains only to prove that $g_n \in \mathfrak{X}^\infty_{\text{fix}}(\mathbb{R}^n)$ for all $n$ sufficiently large, i.e., that $x_f$ is globally asymptotically for $g_n$ for large $n$; this follows from a general result of Smith and Waltman (1999, Thm 2.2).

*Openness.* Let $f \in \mathfrak{X}^\infty_{\text{fix}}(\mathbb{R}^n)$ be a vector field for which $(\mathsf{D}_{x_f}\Phi_f, \mathsf{D}_{x_f}\Phi_f)$ is $\infty$-nonresonant. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence vector fields in $f \in \mathfrak{X}^\infty_{\text{fix}}(\mathbb{R}^n)$ converging to $f$ in the $C^1$ compact-open topology; i.e., $g$ and $\mathsf{D}g$ converge to $f$ and $\mathsf{D}f$ uniformly on compact sets. Since the $C^1$ compact-open topology can be given the structure of a Banach space, the (Banach space version of the) implicit function theorem implies that $x_{g_n} \to x_f$ and hence $\mathsf{D}_{x_{g_n}}\Phi^1_{g_n} = e^{\mathsf{D}_{x_{g_n}} g_n} \to \mathsf{D}_{x_f} f = \mathsf{D}_{x_f}\Phi^1_f$. It now follows from the openness portion of Proposition 13 that $(\mathsf{D}_{x_{g_n}}\Phi_{g_n}, \mathsf{D}_{x_{g_n}}\Phi_{g_n})$ is $\infty$-nonresonant for all $n$ sufficiently large. Since $\mathfrak{X}^\infty_{\text{fix}}$ with the $C^1$ compact-open topology is first countable, this implies the desired openness condition and completes the proof.

Given $f \in \mathfrak{X}^\infty_{\text{per}}(Q))$, we let $x_f$ be an arbitrary point in $\Gamma_f$, $\tau_f$ be the period of $\Gamma_f$, and $E^s_{x_f}$ be the subspace of Proposition 10. The following lemma can be proved using an argument analogous to the one used for Lemma 14.

*Lemma 15.* The subset of vector fields $f \in \mathfrak{X}^\infty_{\text{per}}(Q)$ for which $(\mathsf{D}_{x_f}\Phi^{\tau_f}|_{E^s_{x_f}}, \mathsf{D}_{x_f}\Phi^{\tau_f}|_{E^s_{x_f}})$ is $\infty$-nonresonant is open in $\mathfrak{X}^\infty_{\text{per}}(Q)$ with respect to the $C^1$ compact-open topology, and dense in $\mathfrak{X}^\infty_{\text{per}}(Q)$ with respect to the $C^\infty$ Whitney topology.

Lemmas 14 and 15 immediately imply the following

*Theorem 16.* The subset of vector fields $f \in \mathfrak{X}^\infty_{\text{fix}}(Q)$ for which $\Phi_f$ satisfies all $k = \infty$ hypotheses of Proposition 6 and Theorem 11 is open in $\mathfrak{X}^\infty_{\text{fix}}(Q)$ with respect to the $C^1$ compact-open topology, and dense in $\mathfrak{X}^\infty_{\text{fix}}(Q)$ with respect to the $C^\infty$ Whitney topology.

Similarly, the subset of vector fields $f \in \mathfrak{X}^\infty_{\text{per}}(Q)$ for which $\Phi_f$ satisfies all $k = \infty$ hypotheses of Proposition 10 and Theorem 12 is open in $\mathfrak{X}^\infty_{\text{per}}(Q)$ with respect to the $C^1$ compact-open topology, and dense in $\mathfrak{X}^\infty_{\text{per}}(Q)$ with respect to the $C^\infty$ Whitney topology.

## ACKNOWLEDGEMENTS

We thank George Haller, David Hong, Alexandre Mauroy, Igor Mezić, Jeff Moehlis, Ryan Mohr, Corbinian Schlosser, and Dan Wilson for useful discussions related to this work.

## REFERENCES

de la Llave, R. and Wayne, C.E. (1995). On Irwin's proof of the pseudostable manifold theorem. *Mathematische Zeitschrift*, 219(1), 301–321.

Dellnitz, M. and Junge, O. (1999). On the approximation of complicated dynamical behavior. *SIAM Journal on Numerical Analysis*, 36(2), 491–515.

Fenichel, N. (1974). Asymptotic stability with rate conditions. *Indiana University Mathematics Journal*, 23(12), 1109–1137. doi:10.1090/s0002-9904-1974-13498-1.

Fenichel, N. (1977). Asymptotic stability with rate conditions II. *Indiana University Mathematics Journal*, 26(1), 81–93.

Hirsch, M.W. (1994). *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York. Corrected reprint of the 1976 original.

Hirsch, M.W., Pugh, C.C., and Shub, M. (1977). *Invariant manifolds*. Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin-New York. doi: 10.1007/bfb0092042.

Koopman, B.O. (1931). Hamiltonian systems and transformation in Hilbert space. *Proceedings of the National Academy of Sciences of the United States of America*, 17(5), 315.

Kvalheim, M.D. and Revzen, S. (2019). Existence and uniqueness of global Koopman eigenfunctions for stable fixed points and periodic orbits. *arXiv preprint arXiv:1911.11996*.

Lee, J.M. (2013). *Introduction to Smooth Manifolds*. Springer, 2 edition. doi:10.1007/978-1-4419-9982-5.

Mauroy, A. and Mezić, I. (2016). Global stability analysis using the eigenfunctions of the Koopman operator. *IEEE Transactions on Automatic Control*, 61(11), 3356–3369.

Mauroy, A., Mezić, I., and Moehlis, J. (2013). Isostables, isochrons, and Koopman spectrum for the action–angle representation of stable fixed point dynamics. *Physica D: Nonlinear Phenomena*, 261, 19–30.

Mezić, I. (2005). Spectral properties of dynamical systems, model reduction and decompositions. *Nonlinear Dynamics*, 41(1-3), 309–325.

Mezić, I. and Banaszuk, A. (2004). Comparison of systems with complex behavior. *Physica D: Nonlinear Phenomena*, 197(1-2), 101–133.

Mityagin, B. (2015). The zero set of a real analytic function. *arXiv preprint arXiv:1512.07276*.

Mohr, R. and Mezić, I. (2014). Construction of eigenfunctions for scalar-type operators via Laplace averages with connections to the Koopman operator. *arXiv preprint arXiv:1403.6559*.

Mohr, R. and Mezić, I. (2016). Koopman principle eigenfunctions and linearization of diffeomorphisms. *arXiv preprint arXiv:1611.01209*.

Monga, B., Wilson, D., Matchen, T., and Moehlis, J. (2019). Phase reduction and phase-based optimal control for biological systems: a tutorial. *Biological cybernetics*, 113(1-2), 11–46.

Palis, J. and De Melo, W. (1982). *Geometric theory of dynamical systems: an introduction*. Springer Science & Business Media.

Sell, G.R. (1985). Smooth linearization near a fixed point. *American Journal of Mathematics*, 1035–1091.

Shirasaka, S., Kurebayashi, W., and Nakao, H. (2017). Phase-amplitude reduction of transient dynamics far from attractors for limit-cycling systems. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 27(2), 023119. doi:10.1063/1.4977195.

Smith, H.L. and Waltman, P. (1999). Perturbation of a globally stable steady state. *Proceedings of the American Mathematical Society*, 127(2), 447–453.

Sternberg, S. (1957). Local contractions and a theorem of Poincaré. *American Journal of Mathematics*, 79(4), 809–824.

Wilson, D. and Ermentrout, B. (2018). Greater accuracy and broadened applicability of phase reduction using isostable coordinates. *Journal of mathematical biology*, 76(1-2), 37–66.

Wilson, D. and Moehlis, J. (2016). Isostable reduction of periodic orbits. *Physical Review E*, 94(5), 052213.

Wilson, Jr., F.W. (1967). The structure of the level surfaces of a Lyapunov function. *J. Differential Equations*, 3, 323–329. doi:10.1016/0022-0396(67)90035-6. URL `http://dx.doi.org/10.1016/0022-0396(67)90035-6`.

## Appendix A. NONRESONANT MATRICES HAVE FULL MEASURE

In this appendix we use the fact that all smooth (Hausdorff, paracompact) manifolds have a well-defined notion of *measure zero*, whereby a subset has measure zero if its image in every coordinate chart in some atlas has Lebesgue measure zero in $\mathbb{R}^n$ (Lee, 2013, Ch. 6). (This does not mean that smooth manifolds are equipped with canonical *measures*.)

In the following lemma, recall that a critical point $p \in M$ of a smooth map $f\colon M \to N$ is a point at which $\mathsf{D}_p f\colon \mathsf{T}_p M \to \mathsf{T}_{f(p)} N$ is not surjective.

*Lemma 17.* Let $f\colon M \to N$ be a smooth map between smooth manifolds, and $S \subset N$ a set of measure zero. If the set $C(f) \subset M$ of critical points of $f$ has measure zero, then $f^{-1}(S)$ has measure zero.

**Proof.** Define $M' := M \setminus C(f)$. Since the union of two measure zero sets has measure zero, it suffices to prove that $f^{-1}(S) \cap M'$ has measure zero. By the rank theorem (Lee, 2013, Thm 4.12), for every point $p \in M$ there are neighborhoods $U \ni p$ and $V \ni f(p)$ respectively diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^{m-n}$ and $\mathbb{R}^n$, such that in these coordinates $f(x, y) = x$. The image $S' \subset \mathbb{R}^n$ of $S \cap V$ through the latter diffeomorphism has measure zero, and the image of $f^{-1}(S) \cap U$ in $\mathbb{R}^n \times \mathbb{R}^{m-n}$ through the former diffemorphism is $S' \times \mathbb{R}^m$, so Fubini's theorem implies that $f^{-1}(S) \cap U$ has measure zero (c.f. (Lee, 2013, Lem 6.2)). Since $M'$ is covered by countably many such neighborhoods $U$, $M' \cap f^{-1}(S)$ has measure zero. This completes the proof.

Let $\mathcal{D}_n \subset \mathsf{GL}(n, \mathbb{R})$ denote the set of real invertible matrices having distinct eigenvalues. Let $\mathcal{M}_n \subset \mathbb{C}^n$ be the subset of $n$-vectors whose entries are all distinct, whose non-real entries occur only in complex conjugate pairs, and whose entries are sorted according to the following rules:

- Real eigenvalues appear first, and in increasing order.
- Eigenvalues forming a complex conjugate pair are adjacent, and the eigenvalue having positive imaginary part appears first in each such adjacent pair.
- Complex conjugate pairs are lexicographically ordered first by their real part, then by the absolute value of their imaginary part.

Notice that $\mathcal{M}_n$ is the disjoint union of finitely many open sets $U_1, \ldots, U_\ell$ (where $\ell$ depends on $n$ in a way immaterial for our purposes) defined according to which vector entries have non-zero imaginary parts, and each of the $U_i$ can be identified with an open subset of $\mathbb{R}^n$ in an obvious way. For example, define $U_1$ to be the set of vectors of the form $(a_1, \ldots, a_{n-2}, a_{n-1} + ia_n, a_{n-1} - ia_n)$ with the $a_i \in \mathbb{R}$ ordered according to our rules (so, e.g., $a_n > 0$); we identify such a vector with $(a_1, a_2, a_3, \ldots, a_n) \in \mathbb{R}^n$. Hence $\mathcal{M}_n$ is a real analytic manifold of dimension $n$.

*Proposition 18.* Let $S \subset \mathcal{M}_n$ be a set of measure zero. Then the Lebesgue measure of the set of matrices in $\mathsf{GL}(n, \mathbb{R})$ for which all eigenvalues belong to $S$ has measure zero.

**Proof.** Let $\mathcal{N} \subset \mathsf{GL}(n, \mathbb{R})$ be the set of matrices with eigenvalues belonging to $S$. It is well-known that $\mathsf{GL}(n, \mathbb{R}) \setminus \mathcal{D}_n$ has measure zero, [3] so it suffices to prove that $\mathcal{N} \cap \mathcal{D}_n$ has measure zero.

By construction of $\mathcal{M}_n$, there is a uniquely defined map $\mathcal{E}\colon \mathcal{D}_n \to \mathcal{M}_n$ which sends a matrix in $\mathcal{D}_n$ to its vector of eigenvalues ordered according to the definition of $\mathcal{M}_n$. Since matrices in $\mathcal{D}_n$ have only algebraically simple eigenvalues, and since such eigenvalues depend smoothly on the matrix entries, it follows that $\mathcal{E}$ is smooth. Furthermore, it is readily seen (by writing the matrices in $\mathcal{D}_n$ in real canonical form and considering variations of the block diagonal entries) that $\mathcal{E}$ has no critical points. Thus Lemma 17 implies that $\mathcal{N} \cap \mathcal{D}_n = \mathcal{E}^{-1}(S)$ has measure zero, which completes the proof.

Let $\mathcal{N}_n \subset \mathsf{GL}(n, \mathbb{R})$ be the set of real invertible matrices $A$ *with distinct eigenvalues* such that $(A, A)$ is $\infty$-nonresonant. We want to use Proposition 18 to prove that $\mathsf{GL}(n, \mathbb{R}) \setminus \mathcal{N}_n$ has measure zero. To do this, we need the following lemma.

*Lemma 19.* Let $S_n \subset \mathcal{M}_n \subset \mathbb{C}^n$ be the set of $n$-vectors $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that either (i) $\lambda_j = \lambda_k$ for some $j, k$ or (ii) for some $i \in \{1, \ldots, n\}$ and some $m = (m_1, \ldots, m_n) \in \mathbb{N}^n_{\geq 0}$,

$$\lambda_i - \lambda_1^{m_1} \cdots \lambda_n^{m_n} = 0. \tag{A.1}$$

Then $S_n$ has measure zero in $\mathcal{M}_n$.

**Proof.** Taking the real and imaginary parts of (A.1) yields for each $i, m$ two equations which are polynomial in the real and imaginary parts of the $\lambda_j$. Using the previously described identification of each $U_i$ with an open subset of $\mathbb{R}^n$, we see that each $S_n \cap U_i$ is identified with an open subset of a countable union of zero level sets of non-constant polynomials. Since the zero level set of any non-constant analytic function $\mathbb{R}^n \to \mathbb{R}$ has measure zero, it follows that each $S_n \cap U_i$ has measure zero. Since $S_n = \bigcup_{i=1}^{\ell} (S_n \cap U_\ell)$, it follows that $S_n$ has measure zero in $\mathcal{M}_n$. This completes the proof.

*Corollary 20.* $\mathsf{GL}(n, \mathbb{R}) \setminus \mathcal{N}_n$ has measure zero in $\mathsf{GL}(n, \mathbb{R})$.

**Proof.** Using Definition 2, we see that the set of eigenvalues in $\mathcal{M}_n$ coming from matrices in $\mathsf{GL}(n, \mathbb{R}) \setminus \mathcal{N}_n$ is contained in the set $S_n \subset \mathcal{M}_n$ of Lemma 17. Lemma 17 implies that $S_n$ has measure zero in $\mathcal{M}_n$, so Proposition 18 implies that $\mathsf{GL}(n, \mathbb{R}) \setminus \mathcal{N}_n$ has measure zero in $\mathsf{GL}(n, \mathbb{R})$.

[3] Here is one proof: a matrix has multiple eigenvalues if and only if its discriminant vanishes, the discriminant is a polynomial in the matrix entries, and a level set of an analytic function has measure zero (Mityagin, 2015).