Bioinspired Legged Locomotion

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August 24, 2016

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Chapter 1

Templates and Anchors

1.1 A mathematical framework for legged locomotion

In this section we present a theoretical mathematical framework for analysis and modeling of legged locomotion. This framework is, for most applications, far too general. However, it will serve to provide a precise mathematical foundation inside which other, more practical models and approaches appear as special cases.

The study of legged locomotion is the study of how bodies move through space by deforming appendages we refer to as "legs" and using them to produce reaction forces from the environment that propel the body. Thus, the configuration of the system we seek to study comprises two parts – the location of the body in space, and the "shape" of that body with respect to a frame of reference that travels with the body. In mathematical terms, this means the overall configuration space $\mathcal{Q}$ is:

$$\mathcal{Q} = \mathsf{SE}(3) \times \mathcal{B}, \quad \mathcal{B} \subseteq \mathbb{R}^m \tag{1.1}$$

where $\mathsf{SE}(3)$ is the "Standard Euclidean Group of Dimension 3", also known as "the space of rigid motions", and $\mathcal{B}$ is taken to be some bounded, continuous, closed, piecewise smooth surface in the space $\mathbb{R}^m$. In this chapter, we will typically use $q = (x, b) \in \mathcal{Q}$ to denote the instantaneous configuration.

In this book we are primarily concerned with legged locomotion that is generated by repeating patterns of motion called "gaits". When an animal or robot executes a gait, it traces out a cycle with b in the shape space $\mathcal{B}$, while at the same time translating and/or rotating the body frame $x \in \mathsf{SE}(3)$ through the world. This form of a mathematical structure, in which a space is given by the Cartesian product of a "base space" (in our case, the shape space of the body) and a group[1], here $\mathsf{SE}(3)$, is called a (trivial) "principal fiber bundle"[2], or simply a "principal bundle". Subsets of $\mathcal{Q}$ of the form $\mathsf{SE}(3) \times \{b\}$ for a fixed $b \in \mathcal{B}$ are called "fibers." A very readable introduction to the theory of fiber bundles may be found in Chapter 2 of Bloch et al. [10].

In physics, principal bundles have been used to describe diverse phenomena in which cycles in the base space can be associated with a shift along a fiber. Names for this phenomenon include "Berry Phase", "Geometric Phase", "Pancharatnam Phase" and "non-zero holonomy".

[1] More technically, the group is required to be a "Lie group", and each fiber is a "principal homogeneous space" for this Lie group. Readers interested in these technicalities may consult, for example, Steenrod [74], Husemoller [38].

[2] This bundle is called "trivial" because it is *equal* to a product $\mathsf{SE}(3) \times \mathcal{B}$, whereas in general fiber bundles are spaces which are *locally indistinguishable* from products. See, for example, Bloch et al. [10], Husemoller [38], Steenrod [74].

In the study of locomotion, Geometric Phase has been used to describe the maneuvers cats [48] and geckos [40] use to land on their feet, and the choice of undulatory motions made by snakes and eels [53, 32]. When the relationship between shape change and body frame velocity is linear, it is called a "connection":

$$\dot{x} = A(x, b)\dot{b} \tag{1.2}$$

While technical issues and high dimensions of the models create significant difficulties in applying "geometric mechanics" approaches in practice, this theoretical framework can in principal describe legged systems.

1.2 Templates and anchors: hierarchies of models

One of the most influential insights allowing legged locomotion systems to be analyzed in practice was articulated in Full and Koditschek [24], which proposed the use of "templates" for generating refutable, testable hypotheses for legged locomotion. While a "template" is defined as "*the simplest model (least number of variables and parameters) that exhibits a targeted behavior*", the discussion and more recent treatments of the templates-and-anchors approach follow more closely the concept outlined in Full and Koditschek [24] on page 3329:"*We will say that a more complex dynamic system is an 'anchor' for a simpler dynamic system if (1) motions in its high-dimensional space 'collapse' down to a copy of the lower-dimensional space of motions exhibited by the simpler system and (2) the behavior of the complex system mimics or duplicates that of the simpler system when operating in the relevant (reduced-dimensional copy of) motion space*". In other words, animals have many degrees of freedom, but move "as if" they have only a few, and limit pose to a behaviorally relevant family of postures. From a mathematical perspective, this implies that animals occupy only a low-dimensional "behaviorally relevant" submanifold of $\mathcal{B}$, the space of possible "poses". As an illustrative example of what this means in practice, consider a photo of a galloping horse. We know that the horse is galloping, because the pose ("shape") of the body that we see in that still image is one which is only used for galloping. In fact, there is a cycle of poses that is associated with that horse galloping, and if environmental circumstances contrive to perturb the horse's body away from appropriate shapes for galloping, it quickly returns to some appropriate galloping pose.

However, the insight extends further: trotting quadrupeds such as horses and dogs, running bipeds such as humans and ostriches, insects like cockroaches employing alternating tripod gaits, and even running decapods like ghost crabs all employ similar center of mass dynamics – the "Spring Loaded Inverted Pendulum (SLIP)" [19, 7]. All these organisms exhibit similar center of mass dynamics: each step, they bounce like a pogo stick. The center of mass slows down while descending closer to the ground, reaching its minimum speed at its lowest altitude, while ground reaction force in the normal direction is maximal. The center of mass continues, speeding up as it rises until the body entirely detaches from the ground into an aerial phase of ballistic motion leading to the next step.

In this sense, the SLIP template represents a common governing feature appearing in many organisms when they run quickly. The template is not only a description of a typical subset of poses, but also a low dimensional dynamical model that captures features of the aggregate behavior of the body.

It would be tempting to assume that for every behavior or animal examined, there exists a specific "simplest" template model that governs that behavior. However, the authors of Full and Koditschek [24] had already pointed out that the notion of "simplest" model is problematic, and that both the Lateral Leg Spring (LLS) and the Spring Loaded Inverted Pendulum (SLIP) are templates for running (H_3, H_4 in Full and Koditschek [24] Table 1). The specific formal definition of a "template" was left vague[3].

[3] This was intentional, based on personal communication with each of the authors.

As an illustrative example, both the “Clock-Torque (CT-)SLIP” [70] and “SLIP with knee” [73, 64] models may be considered to be anchors for the classical sagittal plane spring loaded inverted pendulum (SLIP) model; important features of this model can be further distilled (following [7]) into a vertical hopping model, or alternatively into a compass walker [75]. The three-dimensional pogo stick-like SLIP template [71] may also be viewed as an anchor for the sagittal plane SLIP, but one may be equally justified in reducing this three-dimensional pogo stick to a horizontal plane Lateral Leg Spring (LLS) model [67, 68] which captures aspects of the horizontal motion such as steering, but ignores the importance of vertical bouncing. Additionally, both the “hexapedal lateral leg spring” [43] and “jointed lateral leg spring with neurons” [72] models are extensions of the classical LLS which may be viewed as a template for these models. This hierarchy is depicted in Figure 1.1.

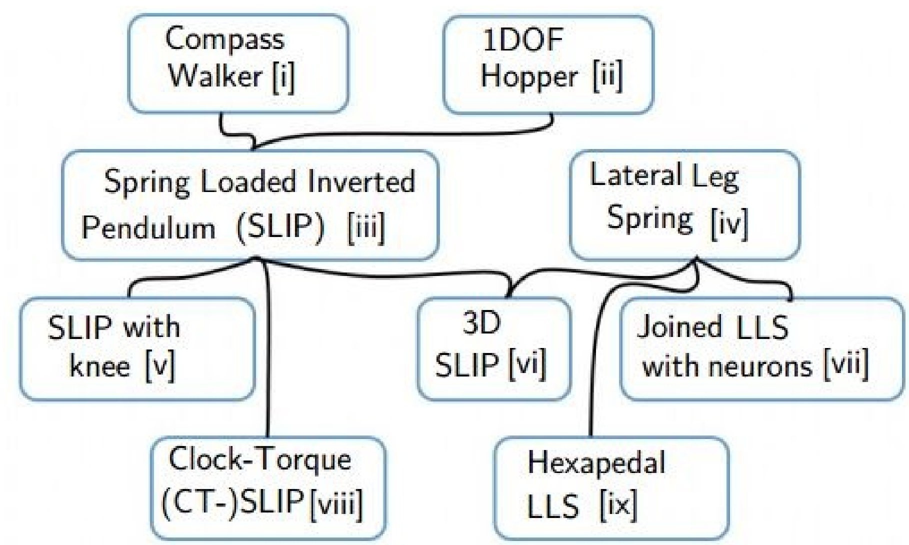

Figure 1.1: A collection of locomotion models with their Template-Anchor relationships indicated, showing a partial order structure. [i] Usherwood et al. [75], [ii] & [iii] Blickhan [7], [iv] Schmitt and Holmes [67, 68], [v] Rummel and Seyfarth [64], Seyfarth et al. [73], [vi] Seipel and Holmes [71] [vii] Seipel et al. [72], [viii] Seipel and Holmes [70], [ix] Kukillaya and Holmes [43]

We are led to the conclusion that rather than a template being a unique, ultimate object, “template and anchor” is a relationship between models. A given model Y can be a template for a more anchored model X, while Y itself may be an anchor for a further template Z. We will use the term “template” to imply that this model is “simpler” than its “anchor”. Usually, one aspect of this simplicity is a reduction of dimension, and quantities in a template often represent aggregates of quantities from the underlying anchor. For example, both SLIP and LLS reduce the mass distribution of the body to a concentrated mass with or without rotational inertia; both discard modeling the kinetic energy and momentum associated with the legs themselves. An insightful discussion on the design and control of legged robots using template-anchor notions is given in Blickhan et al. [8].

In the remainder of this chapter we will discuss several of the ways in which a template-and-anchor hierarchy can be constructed to facilitate the understanding of legged locomotion.

As a cautionary note, it should be pointed out that the term “template” has sometimes been used to mean “a spring mass model of center of mass dynamics”. In this book, we will use it in the much broader meaning described above.

1.3 Templates in dynamics, control, and modeling

There are several ways to approach template-anchor relationships which have been used successfully. Mathematicians and physicists studying dynamical systems theory have constructed a variety of notions of dimensionality reduction. From this perspective, the primary object of study is an elaborate mathematical “anchor” model comprising a set of equations, the solutions of which are shown to be approximately or exactly modeled by a simpler “template” model comprising fewer equations with fewer parameters. Several examples of this approach can be found in [35]. In particular, Kukillaya and Holmes [43] have shown an example of a hexapedal cockroach model with jointed legs and neuronal control, which can be formally reduced and shown to behave similarly to the far simpler LLS model.

Engineers building robots have looked to templates as “targets of control”, i.e. as descriptions of desirable behaviors to be emulated [78, 60, 2], or as simplifications to be used for quickly estimating an appropriate control policy [56]. Here, the primary object of study is not the template itself, as much as it is the means by which template dynamics are elicited from an anchor.

A more explicit focus on templates is found in work by engineers employing “template-based” strategies for the bio-inspired design and control of robots. Here, the goal is to embed well-known templates in more complex, anchored locomotion systems. Controllers have been designed to embed the dynamics encoded in SLIP and its three-dimensional analog in bipedal robots [77, 17]. Ankarali and Saranli [4] have used an extended SLIP model involving torque actuation at the hip for designing a controller to achieve underactuated planar pronking in the robot RHex [65]. Other researchers have considered the combination of several different templates in the same robot in order to render it capable of achieving multiple goals, such as running/climbing [51] and running/reorientation/vertical hopping [18]. Lee et al. [44] have even worked to embed a cockroach-inspired antenna-based wall-following template in a robot with a bio-inspired antenna.

Biomechanists have looked to templates from a data-driven, experiment-centric perspective. Here, the primary objects of study are the locomotion data themselves. The goal is to find low-dimensional models which represent observational data, accounting for both trends and variability with a few meaningful parameters. Data-driven templates have been used successfully to predict how animals recover from perturbations [61] and how humans control and stabilize their running gait [50]. These ideas are elaborated upon in Section 1.4.3.

1.4 Sources of templates; notions of templates

Given the three approaches to templates described in the previous section, it is hardly surprising that there are many mathematical notions of being a template-and-anchor pair. In this section we point to some of the literature in the field. The subtle differences and technical caveats associated with applying these notions are outside the scope of our exposition.

1.4.1 Dimensionality reduction in dynamical systems

As a simple example, templates exist in the dynamics of linear systems (see, e.g., the textbook Hirsch and Smale [33] for an introduction to linear systems). When a stable Linear Time Invariant (LTI) system of differential equations $\dot{x} = Ax$ has a large “spectral gap” – some modes (projections of solutions onto the generalized eigenspaces of A) collapse much faster than others – the slower modes can justifiably be viewed as a template for the complete higher dimensional system. This expresses itself as a large difference in the real part of the eigenvalues of the matrix A, with the eigenvalues corresponding to slow template modes having a real part close to zero.

Dynamicists have extended this idea to nonlinear systems in multiple ways using the notion of "invariant manifolds", of which the generalized eigenspaces in the previous example are a special case. A positively (negatively) invariant manifold is a smooth submanifold of the state space of a dynamical system for which any initial condition belonging to this submanifold remains in the submanifold as it evolves forward (backward) in time. An invariant manifold is a smooth submanifold of the state space of a dynamical system which is both negatively and positively invariant; in other words, an invariant manifold is a union of trajectories. (Positively) invariant manifolds are often useful notions of templates – here, the template appears in a form which guarantees that the anchor dynamics restricted to template states are invariant, meaning that if the anchor begins in a state belonging to the template it can no longer escape back to exhibiting more complex behaviors. An excellent survey of the many ways invariant manifold methods have been useful in science and engineering is given in Chapter 1 of Wiggins [79].

One well-known class of invariant manifolds which can be used to form useful templates are the normally hyperbolic invariant manifolds (NHIMs) [34, 21, 79] which are asymptotically stable in the sense that they attract all nearby trajectories asymptotically. Special cases of NHIMs include hyperbolic fixed points and hyperbolic periodic orbits [33]. A particularly nice property of NHIMs is that they persist under small smooth perturbations of the equations defining the dynamical system [34], and the compact invariant manifolds which persist under smooth perturbations are normally hyperbolic [47]. This makes NHIMs useful from a modeling perspective. Since physical measurements cannot determine parameters of a mathematical model with perfect accuracy, any physically meaningful feature of a mathematical model must persist under small perturbations.

Viewed as infinite-dimensional dynamical systems, even certain partial differential equations admit a template-like structure both through the theory of normal hyperbolicity [6, 5] and the related theory of "inertial manifolds," the second class of (positively) invariant manifolds we mention here [15, 23]. Inertial manifolds, when they exist, are finite-dimensional positively invariant manifolds containing the global attractor of a (possibly infinite-dimensional) dynamical system and attracting all solutions at an exponential rate [23]. If an inertial manifold exists for a given partial differential equation, it governs the long-term dynamics. Examples of systems having an inertial manifold include dissipative systems such as those that appear in elasticity and fluid dynamics [15]. Techniques for computationally producing approximate inertial manifolds have been studied [22].

"Center manifolds" are the third class of invariant manifolds we mention here. We briefly describe the most basic notion of center manifold at the level of generality relevant for our discussion; see, for example, the discussion in Section 3.2 of [30] for more details. Given a system of differential equations $\dot{x} = f(x)$ and a stable equilibrium point x_0 with $f(x_0) = 0$, the eigenvalues of the linearization $\mathsf{D}f(x_0)$ split into collections of eigenvalues having negative and zero real part. These collections of eigenvalues respectively determine stable and center subspaces. The center manifold theorem states that there exist "stable" and "center" invariant manifolds respectively tangent to these subspaces. Trajectories in the stable manifold approach x_0 exponentially in positive time. While the stable and manifold are always unique, in general the center manifold need not be. Center manifolds may also be defined for periodic orbits (see Theorem 4 of section 3.5 in [55]) and more general attractors [14]. Center manifolds and NHIMs have somewhat similar spectral properties, but they differ in that NHIMs have an instrinsic global definition whereas center manifolds are only defined locally. This local definition manifests itself in the fact that center manifolds are in general nonunique (see section 1.1.2 of [21] for more discussion).

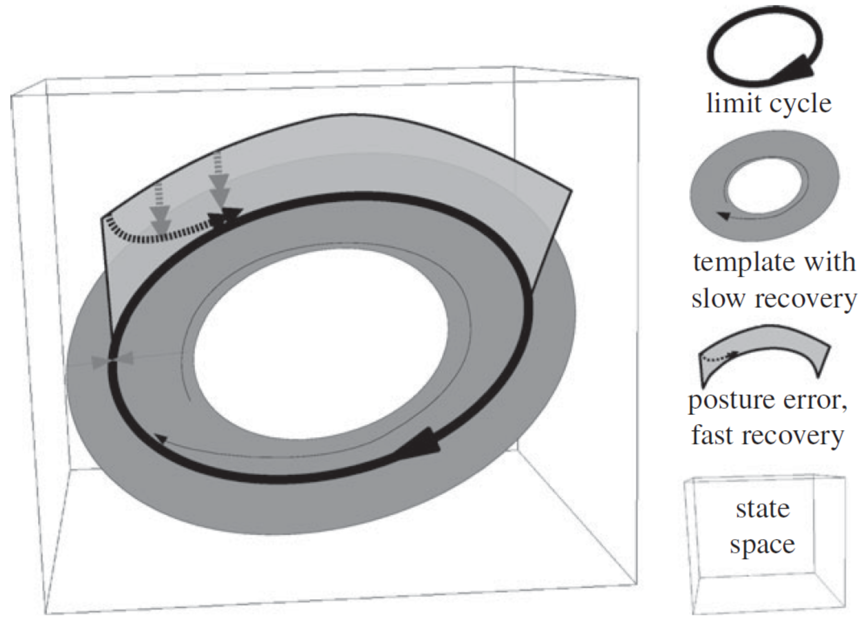

Figure 1.2: An example of an invariant manifold template-anchor relationship in the context of modeling legged locomotion shape-space dynamics by an oscillator. The collection of states corresponding to Floquet multipliers with relatively large magnitude form an invariant "slow manifold". The states belonging to this invariant manifold may be thought of as the states which return slowly to an unperturbed gait, modeled by the limit cycle. Taking the anchor to be the dynamics on the entire state space, the dynamics restricted to this invariant slow manifold may serve as a template. Alternatively, the dynamics restricted to the states traced out by the limit cycle itself may serve as a "phase oscillator" template which is a coarser approximation of the anchor dynamics.

All of the classes of (positively) invariant manifolds we have mentioned have the property that they attract all nearby states. The template is stable in the sense that anchor states which are near template states will asymptotically approach the template. However, one important reason these notions of templates are so useful is more subtle than this; not only do nearby anchor states approach these invariant manifold templates, they approach *specific trajectories* in the template. This provides justification for the approximation of anchor dynamics by template dynamics. For inertial manifolds, this property is known in the literature as "asymptotic completeness" [62], and the fact that center manifolds have this property is shown, for example, in Carr [13]. For NHIMs, this property is often noted by referring to the existence of an "invariant foliation" or "invariant fibration" of the basin of attraction of the invariant manifold [34], and is also sometimes referred to as "asymptotic phase" in the literature [11] (we refer to this as "dynamical phase" in Chapter **Internal reference to oscillator chapter**.). Guckenheimer [29] contains a simpler discussion of the properties of asymptotic phase for the special case of exponentially stable limit cycles.

As an illustrative example of the utility of invariant manifold notions of templates in the analysis of legged locomotion, consider an oscillator. As explained in Chapter **Internal reference to oscillator chapter**, an oscillator, by definition, consists of the dynamics in the basin of attraction of an exponentially stable periodic orbit (also known as a limit cycle)[4]. The image of the periodic orbit, or set of points traced out by the limit cycle, is itself a normally hyperbolic invariant manifold. Defining the anchor to be the dynamics on the entire basin of attraction, a template may be taken to consist of the dynamics restricted to the image of the periodic orbit. Explicitly, the existence of asymptotic phase on the basin of attraction implies that each anchor state will asymptotically coalesce with a (in this case, unique) template state which may be represented by assigning to each anchor state a number $\theta \in [0, 2\pi)$. This is the "phase oscillator" template explained in Chapter **Internal reference to oscillator chapter**. However, for many practical applications, this particular template approximation of the anchor dynamics may be too coarse. As explained in Chapter **Internal reference to oscillator chapter**, the theory of normal forms [11] shows that large "spectral gaps" in the Floquet multipliers of an oscillator yield additional invariant slow manifolds corresponding to slow Floquet modes. Anchor states will again asymptotically approach particular template trajectories in such a way that the dynamics restricted to this invariant manifold constitutes a good template approximation of the anchor dynamics. Physically, the limit cycle may be viewed as representing a perfectly periodic gait subject to no environmental or neuromuscular perturbations. The invariant slow manifold template may then be viewed as the collection of anchor states having "slow recovery" when perturbed from this steady gait. Any anchor states not belonging to this template will quickly return to the template and may be viewed as "posture errors". This is illustrated in Figure 1.2.

Yet another source of templates comes from mechanical models possessing symmetries. Roughly speaking, a differential equation is said to possess a "symmetry" if it is invariant under the action of a Lie group [46] on its state space. For the case of mechanical systems, reduction tools such as Noether's Theorem from geometric mechanics [1, 9] yield conserved quantities (e.g. energy, momentum, angular momentum) which constrain trajectories of the dynamical system to lower-dimensional submanifolds. Dynamics restricted to these lower-dimensional submanifolds form templates for the original anchored mechanical system, and one can understand the behavior of the template in terms of the anchor and vice-versa. We note that other reduction methods in the spirit of Galois' work on algebraic equations [20] also exist for the analysis of ordinary differential equations possessing symmetries but not arising from mechanical systems [52].

An even greater focus on templates as approximations can be found in the theory of "bisimulation" appearing in its original form in the study of discrete state transition systems in computer science [54]. Intuitively, two systems are bisimilar if they cannot be distinguished by an external observer. Bisimulation has been generalized to apply to continuous-time and hybrid dynamical systems. In fact, the template notions previously mentioned in this section are bisimulations of their anchor dynamics, which follows from Proposition 11 of Haghverdi et al. [31]. Bisimulation provides a formalism for discussing templates and anchors for situations more general than the case in which the template is an invariant submanifold of the anchor.

Despite the level of generality afforded by the framework of bisimulation, requiring bisimilarity between models as a criterion for template-anchor relationships can sometimes arguably be too restrictive for modeling physical systems. Bisimilarity relations are not robust to noise, measurement error, or other perturbations to physical models. Recent work has extended the notion of bisimilarity by providing a definition of "approximate bisimulation" [26]. The utility of approximate bisimulation lies in its ability to quantify the quality of approximation by one mathematical model of another. In particular, the language of approximate bisimulation can be used to quantify the degree to which some mathematical model is a template for another anchored model. As a simple example, a double pendulum with one small mass m and one large mass M can be approximated by a single pendulum of mass $M + m$. For a more interesting example, consider the following. Animals, such as dogs, are able to instantiate the same template despite seemingly catastrophic

[4]Recall from Chapter **Internal reference to oscillator chapter** that an "oscillator" is a *deterministic* system as defined here, while in Chapter **Internal reference to oscillator chapter** we used the term "rhythmic system" to refer to a *nondeterministic* system resulting from perturbation of a (deterministic) oscillator by (relatively small) noise.

injury such as limb loss. Figure 1.3 illustrates the approximate template-anchor relationships between relevant models for this case. These examples are hardly surprising from the perspective of mechanical intuition, but the theory of approximate bisimulation renders these observations formal, testable and quantifiable in a computational framework.

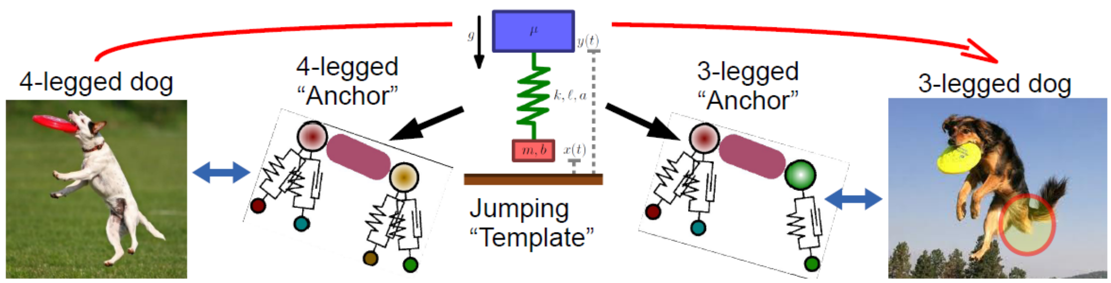

Figure 1.3: An uninjured dog and a 3-legged dog can both jump to catch a frisbee. The ability to do so well can be expressed (red arrow) by taking the dynamics anchored in the 4-legged morphology, abstracting them as a jumping template and embodying this template in a 3-legged anchor reflecting the new morphology. The quality of this abstraction and embodiment can be quantified in a formal way within the framework of approximate bisimulation.

1.4.2 Templates based on mechanical intuition

By far, the most prolific source of models intended as templates has been the insight of researchers. As described in Section 1.2, the insight of Blickhan led to the introduction of the SLIP template in his seminal work [7]. Despite being an energetically conservative model without control inputs, the SLIP has enjoyed enormous success in making tractable the tasks of animal locomotion analysis (Section 1.2) and robot design and control (Section 1.3). The success of the sagittal plane spring-loaded inverted pendulum as a mathematical model inspired various extensions of SLIP, such as CT-SLIP [70], as well as three-dimensional [71], bipedal [25], and segmented versions of SLIP [73, 64]. Other templates such as LLS [67, 68] and its extensions (described in Section 1.2) such as a model with additional joints and neuronal interactions [72] and a hexapedal version of LLS [43] were developed to specifically capture the horizontal component of locomotion. Thoughtful consideration of modeling has produced a plethora of additional templates of varying levels of complexity appropriate for other situations. Inspired by the climbing aptitude of insects and geckos, Goldman et al. [28] proposed a template for describing rapid vertical climbing. Observations of cockroaches using their antennae to follow walls motivated the introduction of an antenna-based wall following template [16]. Human walking inspired a template based on the notion of "virtual pivot points" [49]. Examples of other templates proposed for specific classes of models include a quadrupedal running template for robotic systems with articulated torsos [12] and a kinematic template proposed for an eight-legged miniature octopedal robot assumed to be in quasi-static motion [41].

Typically, templates have been proposed without explicitly formulating the anchor model to which they relate (e.g. [12, 28]), although there are exceptions (e.g. [41]). In other work, there is an emphasis placed on exploring relationship between various templates and their anchors. To name but a few examples, see Seipel and Holmes [71, 69], as well as Chapter 5 of [35] and the references therein.

1.4.3 Data-driven model reduction

Data-driven dimensionality and model reduction reduction has emerged as an industrious and interdisciplinary field of research, having broad applications to the science and engineering fields and drawing upon techniques from optimization, statistics, dynamical systems theory, and machine learning. Classical approaches to dimensionality reduction include linear subspace projection methods such as Principal Component Analysis and Factor Analysis [39]; an active area of current research is on nonlinear dimensionality reduction approaches such as Manifold Learning [45], which generalize linear projection methods by replacing linear subspaces with submanifolds. Projection methods such as these identify a small (relative to the dimensionality of the raw data) collection of parameters which may accurately represent the raw data, and this collection of parameters is optimal in some sense depending on the projection method used. Such a small parameter set may accurately capture the spatial information present in time series data and motivate the construction of reduced-order spatio-temporal mathematical models. Givon et al. [27] contains a review of several other algorithmic approaches to dimensionality reduction, focusing on methods specifically aimed at model reduction of general dynamical systems.

In the context of legged locomotion, there has been work on the construction of templates directly motivated from data. Under the assumption that the underlying mathematical model is an oscillator (see Chapter **Internal reference to oscillator chapter**), several researchers have performed nonparametric system identification of biomechanical systems [3, 76, 57, 36, 37]. Researchers have additionally attempted to find nonlinear coordinate systems directly from data in which oscillator dynamics are linear (see Revzen and Kvalheim [59] and references therein for more mathematical detail), and have coined the term "Data-Driven Floquet Analysis" (DDFA) to collectively refer to the computation of this linearizing coordinate system and to other oscillator system identification methods [57]. The linearizing coordinate change of DDFA can be viewed as a special case of finding linearizing "observables", which are themselves eigenfunctions of the Koopman operator [63, 42], and may in some cases be computed using Dynamic Mode Decomposition [66] and its extensions. Using the techniques of DDFA, Revzen and Guckenheimer [58] present a method for identifying appropriate dimensions of reduced-order models of legged locomotion and other rhythmic systems directly from noisy data and without explicit knowledge of governing equations. By exploiting the topological structure of the stability basin of an oscillator, they determine a candidate dimension for the slow manifold by examining the magnitudes of eigenvalues of Poincaré return maps. This candidate dimension serves as an upper bound for the dimension of a statistically significant template.

In the specific context of human walking, Wang used analysis of Poincaré maps to show a relationship between upper body/trunk motion and foot placements, providing a rigorous data-driven derivation of human walking features previously conjectured [76]. Maus et al. [50] performed DDFA on human running data and show that while the SLIP template predicts within-step kinematics of the center of mass, it fails to predict stability and behavior beyond one step. Furthermore, insights derived from DDFA enable Maus et al. [50] to identify that swing-leg ankle states are important predictors of human locomotion beyond those present in the SLIP template. Augmenting the SLIP model with these predictors, the authors construct a model shown to have predictive power superior to SLIP for the available subject population.

1.5 Conclusion

The answer to the question of "which notion of template-anchor relationship to use?" depends on one's goals and on practical limitations of the application in mind.

As an example of one end of the spectrum, mathematicians wanting to explore mathematical relationships need to write down or use existing equations of motion, which may make many assumptions about the underlying physics and/or biology of a locomoting system. In this case, various templates may be amenable to discovery by theoretical consideration. For example, invariant manifolds may be found "by hand" or numerical methods. Alternatively, reduction tools from geometric mechanics and the theory of Lie groups may be used to produce templates if symmetries are present in the equations of motion. Notions such as bisimulation and approximate bisimulation from computer science are used to formalize such template-anchor notions in some areas of the literature.

On the opposite end of the spectrum, experimental biologists deal with actual data and do not have access to explicit mathematical models *a priori*. For this reason, researchers have worked on data-driven methods of system identification and model reduction. As outlined in Section 1.4.3, many algorithms have been used in attempts to tackle this problem for real-world systems in general, and several researchers have worked on methods aimed specifically towards legged locomotion. In particular, there has been some success in using Data Driven Floquet Analysis both using data to directly explain previously conjectured features of human locomotion and in motivating new templates of human running which may outperform SLIP as predictive models.

In between these two extremes, engineers and control theorists need methods to obtain practical models amenable to computation for which they can produce their own template "targets of control" to achieve desirable behaviors in a robotic system. Many engineers have used "template-based" methods in the bio-inspired design and control of robots, attempting directly to embed the low-dimensional dynamics classical templates such as SLIP in high-dimensional anchored robots in order to achieve useful behaviors.

There are a myriad of notions and examples of "templates and anchors" outlined in this chapter. Many of the approaches appear quite distinct, but many engineers, scientists, and mathematicians may benefit from exposure to new ways of thinking about old ideas.

This work was funded in part by ARO grant #W911NF-14-1-0573 "Morphologically Modulated Dynamics" to S. Revzen.

Bibliography

[1] R Abraham and JE Marsden. *Foundations of mechanics*. Benjamin/Cummings Publishing Company Reading, Massachusetts, 1978.

[2] Aaron D Ames. Human-inspired control of bipedal walking robots. *Automatic Control, IEEE Transactions on*, 59(5):1115–1130, 2014. doi: 10.1109/TAC.2014.2299342.

[3] MM Ankarali. *Variability, symmetry, and dynamics in human rhythmic motor control*. PhD thesis, PhD thesis, Johns Hopkins University, Baltimore, MD, USA, 2015.

[4] MM Ankarali and U Saranli. Control of underactuated planar pronking through an embedded spring-mass hopper template. *Autonomous Robots*, 30(2):217–231, 2011. doi: 10.1007/s10514-010-9216-x.

[5] P Bates, K Lu, and C Zeng. Invariant foliations near normally hyperbolic invariant manifolds for semiflows. *Transactions of the American Mathematical Society*, 352(10):4641–4676, 2000. doi: 10.1090/S0002-9947-00-02503-4.

[6] PW Bates, K Lu, and C Zeng. *Existence and persistence of invariant manifolds for semiflows in Banach space*, volume 645. American Mathematical Soc., 1998.

[7] R Blickhan. The spring-mass model for running and hopping. *Journal of biomechanics*, 22(11-12):1217–1227, 1989. doi: 10.1016/0021-9290(89)90224-8.

[8] R Blickhan, A Seyfarth, H Geyer, S Grimmer, H Wagner, and M Gunther. Intelligence by mechanics. *Philos Trans R Soc Lond , Ser A: Math , Phys Eng Sci*, 365(1850):199–220, January 2007.

[9] AM Bloch, PS Krishnaprasad, JE Marsden, and RM Murray. Nonholonomic mechanical systems with symmetry. *Archive for Rational Mechanics and Analysis*, 136(1):21–99, 1996.

[10] AM Bloch, J Baillieul, P Crouch, JE Marsden, PS Krishnaprasad, RM Murray, and D Zenkov. *Nonholonomic mechanics and control*, volume 24. Springer, 2003.

[11] AU Bronstein and AY Kopanskii. *Smooth Invariant Manifolds and Normal Forms*. World Scientific Publishing, Salem, Massachusetts, 1 edition, 1994. ISBN 981021572X.

[12] Q Cao and I Poulakakis. Quadrupedal bounding with a segmented flexible torso: passive stability and feedback control. *Bioinspiration & biomimetics*, 8(4):046007, 2013. doi: 10.1088/1748-3182/8/4/046007.

[13] J Carr. *Applications of centre manifold theory*, volume 35. Springer Science & Business Media, 1982.

[14] S-N Chow, W Liu, and Y Yi. Center manifolds for smooth invariant manifolds. *Transactions of the American Mathematical Society*, 352(11):5179–5211, 2000. doi: 10.1090/S0002-9947-00-02443-0.

[15] P Constantin, C Foias, B Nicolaenko, and R Temam. *Integral manifolds and inertial manifolds for dissipative partial differential equations*, volume 70. Springer Science & Business Media, 2012.

[16] NJ Cowan, J Lee, and RJ Full. Task-level control of rapid wall following in the american cockroach. *Journal of Experimental Biology*, 209(9):1617–1629, 2006. doi: 10.1242/jeb.02166.

[17] B Dadashzadeh, HR Vejdani, and J Hurst. From template to anchor: A novel control strategy for spring-mass running of bipedal robots. In *Intelligent Robots and Systems (IROS 2014), 2014 IEEE/RSJ International Conference on*, pages 2566–2571. IEEE, 2014.

[18] A De and DE Koditschek. Parallel composition of templates for tail-energized planar hopping. In *Robotics and Automation (ICRA), 2015 IEEE International Conference on*, pages 4562–4569. IEEE, 2015. doi: 10.1109/ICRA.2015.7139831.

[19] MH Dickinson, CT Farley, RJ Full, MAR Koehl, R Kram, and S Lehman. How animals move: an integrative view. *Science*, 288(5463):100–106, 2000.

[20] DS Dummit and RM Foote. *Abstract algebra*, volume 1984. Wiley Hoboken, 2004.

[21] J Eldering. Normally hyperbolic invariant manifolds. *Atlantis Studies in Dynamical Systems*, 2, 2013. doi: 10.2991/978-94-6239-003-4.

[22] C Foias, MS Jolly, IG Kevrekidis, GR Sell, and ES Titi. On the computation of inertial manifolds. *Physics Letters A*, 131(7):433–436, 1988. doi: 10.1016/0375-9601(88)90295-2.

[23] C Foias, GR Sell, and R Temam. Inertial manifolds for nonlinear evolutionary equations. *Journal of Differential Equations*, 73(2):309–353, 1988. doi: 10.1016/0022-0396(88)90110-6.

[24] RJ Full and DE Koditschek. Templates and anchors: neuromechanical hypotheses of legged locomotion on land. *Journal of Experimental Biology*, 202(23):3325–3332, 1999.

[25] H Geyer, A Seyfarth, and R Blickhan. Compliant leg behaviour explains basic dynamics of walking and running. *Proceedings of the Royal Society of London B: Biological Sciences*, 273(1603):2861–2867, 2006. doi: 10.1098/rspb.2006.3637.

[26] A Girard and GJ Pappas. Approximation metrics for discrete and continuous systems. *Automatic Control, IEEE Transactions on*, 52(5):782–798, 2007. doi: 10.1109/TAC.2007.895849.

[27] D Givon, R Kupferman, and A Stuart. Extracting macroscopic dynamics: model problems and algorithms. *Nonlinearity*, 17(6):R55, 2004.

[28] DI Goldman, TS Chen, DM Dudek, and RJ Full. Dynamics of rapid vertical climbing in cockroaches reveals a template. *Journal of Experimental Biology*, 209(15):2990–3000, 2006. doi: 10.1242/jeb.02322.

[29] JM Guckenheimer. Isochrons and phaseless sets. *Journal of Mathematical Biology*, 1:259–273, 1975. doi: 10.1007/BF01273747.

[30] JM Guckenheimer. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer-Verlag, New York, New York, 1 edition, 1983. ISBN 0387908196.

[31] E Haghverdi, P Tabuada, and GJ Pappas. Bisimulation relations for dynamical, control, and hybrid systems. *Theoretical Computer Science*, 342(2):229–261, 2005. doi: 10.1016/j.tcs.2005.03.045.

[32] RL Hatton and H Choset. Geometric motion planning: The local connection, stokes theorem, and the importance of coordinate choice. *The International Journal of Robotics Research*, 30(8):988–1014, 2011.

[33] MW Hirsch and S Smale. *Differential Equations, Dynamical Systems, and Linear Algebra*. Academic Press, New York, New York, 1 edition, 1974. ISBN 0123495504.

[34] MW Hirsch, CC Pugh, and M Shub. Invariant manifolds. *Bull. Amer. Math. Soc.*, 76(5):1015–1019, 09 1970.

[35] P Holmes, RJ Full, DE Koditschek, and JM Gukenheimer. The dynamics of legged locomotion : models, analyses, and challenges. *SIAM Review*, 48(2):206–304, 2006. doi: 10.1137/S0036144504445133.

[36] Y Hurmuzlu and C Basdogan. On the measurement of dynamic stability of human locomotion. *Journal of biomechanical engineering*, 116(1):30–36, 1994. doi: 10.1115/1.2895701.

[37] Y Hurmuzlu, C Basdogan, and D Stoianovici. Kinematics and dynamic stability of the locomotion of post-polio patients. *Journal of biomechanical engineering*, 118(3):405–411, 1996. doi: 10.1115/1.2796024.

[38] D Husemoller. Fibre bundles, volume 20 of graduate texts in mathematics, 1994.

[39] I Jolliffe. *Principal component analysis*. Wiley Online Library, 2002. doi: 10.1002/9781118445112.stat06472.

[40] A Jusufi, DI Goldman, S Revzen, and RJ Full. Active tails enhance arboreal acrobatics in geckos. *Proceedings of the National Academy of Sciences*, 105(11):4215–4219, 2008.

[41] K Karydis, Y Liu, I Poulakakis, and HG Tanner. A template candidate for miniature legged robots in quasi-static motion. *Autonomous Robots*, 38(2):193–209, 2015. doi: 10.1007/s10514-014-9401-4.

[42] BO Koopman. Hamiltonian systems and transformation in hilbert space. *Proceedings of the National Academy of Sciences of the United States of America*, 17(5):315, 1931.

[43] RP Kukillaya and PJ Holmes. A hexapedal jointed-leg model for insect locomotion in the horizontal plane. *Biological cybernetics*, 97(5-6):379–395, 2007. doi: 10.1007/s00422-007-0180-2.

[44] J Lee, SN Sponberg, OY Loh, AG Lamperski, RJ Full, and NJ Cowan. Templates and anchors for antenna-based wall following in cockroaches and robots. *Robotics, IEEE Transactions on*, 24(1):130–143, 2008. doi: 10.1109/TRO.2007.913981.

[45] JA Lee and M Verleysen. *Nonlinear dimensionality reduction*. Springer Science & Business Media, 2007.

[46] JM Lee. Smooth manifolds. In *Introduction to Smooth Manifolds*, pages 1–31. Springer, 2013. doi: 10.1007/978-1-4419-9982-5_1.

[47] R Mané. Persistent manifolds are normally hyperbolic. *Transactions of the American Mathematical Society*, 246:261–283, 1978. doi: 10.1090/S0002-9947-1978-0515539-0.

[48] JE Marsden, OM O'Reilly, FJ Wicklin, and BW Zombros. Symmetry, stability, geometric phases, and mechanical integrators. *Nonlinear Science Today*, 1(1):4–11, 1991.

[49] H-M Maus, SW Lipfert, M Gross, J Rummel, and A Seyfarth. Upright human gait did not provide a major mechanical challenge for our ancestors. *Nature communications*, 1:70, 2010. doi: 10.1038/ncomms1073.

[50] H-M Maus, S Revzen, JM Guckenheimer, C Ludwig, J Reger, and A Seyfarth. Constructing predictive models of human running. *Journal of the Royal Society Interface*, 12, 2014. doi: 10.1098/rsif.2014.0899.

[51] BD Miller and JE Clark. Towards highly-tuned mobility in multiple domains with a dynamical legged platform. *Bioinspiration & biomimetics*, 10(4):046001, 2015. doi: 10.1088/1748-3190/10/4/046001.

[52] PJ Olver. *Applications of Lie groups to differential equations*, volume 107. Springer Science & Business Media, 2000.

[53] J Ostrowski and J Burdick. The geometric mechanics of undulatory robotic locomotion. *The international journal of robotics research*, 17(7):683–701, 1998.

[54] D Park. *Concurrency and automata on infinite sequences*. Springer, 1981. doi: 10.1007/BFb0017309.

[55] L Perko. *Differential Equations and Dynamical Systems*. Springer-Verlag, New York, New York, 3 edition, 2001. ISBN 0387951164.

[56] MH Raibert, HB Brown, and M Chepponis. Experiments in balance with a 3d one-legged hopping machine. *The International Journal of Robotics Research*, 3(2):75–92, 1984. doi: 10.1177/027836498400300207.

[57] S Revzen. *Neuromechanical Control Architectures of Arthropod Locomotion*. PhD thesis, University of California, Berkeley, 2009.

[58] S Revzen and JM Guckenheimer. Finding the dimension of slow dynamics in a rhythmic system. *J. R. Soc Interface*, 2011. doi: 10.1098/rsif.2011.043.

[59] S Revzen and M Kvalheim. Data driven models of legged locomotion. In *SPIE Defense+ Security*, pages 94671V–94671V. International Society for Optics and Photonics, 2015. doi: 10.1117/12.2178007.

[60] S Revzen, BD Ilhan, and DE Koditschek. Dynamical trajectory replanning for uncertain environments. In *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*, pages 3476–3483. IEEE, 2012. doi: 10.1109/CDC.2012.6425897.

[61] S Revzen, SA Burden, TY Moore, J-M Mongeau, and RJ Full. Instantaneous kinematic phase reflects neuromechanical response to lateral perturbations of running cockroaches. *Biological cybernetics*, 107(2):179–200, 2013. doi: 10.1007/s00422-012-0545-z.

[62] JC Robinson. The asymptotic completeness of inertial manifolds. *Nonlinearity*, 9(5):1325, 1996.

[63] CW Rowley, I Mezić, S Bagheri, P Schlatter, and DS Henningson. Spectral analysis of nonlinear flows. *Journal of fluid mechanics*, 641:115–127, 2009. doi: 10.1017/S0022112009992059.

[64] J Rummel and A Seyfarth. Stable running with segmented legs. *The International Journal of Robotics Research*, 27(8):919–934, 2008. doi: 10.1177/0278364908095136.

[65] U Saranli, M Buehler, and DE Koditschek. Rhex: A simple and highly mobile hexapod robot. *The International Journal of Robotics Research*, 20(7):616–631, 2001. doi: 10.1177/02783640122067570.

[66] PJ Schmid. Dynamic mode decomposition of numerical and experimental data. *Journal of fluid mechanics*, 656:5–28, 2010. doi: 10.1017/S0022112010001217.

[67] J Schmitt and P Holmes. Mechanical models for insect locomotion: dynamics and stability in the horizontal plane i. theory. *Biological cybernetics*, 83(6):501–515, 2000. doi: 10.1007/s004220000181.

[68] J Schmitt and P Holmes. Mechanical models for insect locomotion: dynamics and stability in the horizontal plane–ii. application. *Biological cybernetics*, 83(6):517–527, 2000. doi: 10.1007/s004220000180.

[69] J Seipel and P Holmes. Three-dimensional translational dynamics and stability of multi-legged runners. *The International Journal of Robotics Research*, 25(9):889–902, 2006. doi: 10.1177/0278364906069045.

[70] J Seipel and P Holmes. A simple model for clock-actuated legged locomotion. *Regular and chaotic dynamics*, 12(5):502–520, 2007. doi: 10.1134/S1560354707050048.

[71] JE Seipel and P Holmes. Running in three dimensions: Analysis of a point-mass sprung-leg model. *The International Journal of Robotics Research*, 24(8):657–674, 2005. doi: 10.1177/0278364905056194.

[72] JE Seipel, PJ Holmes, and RJ Full. Dynamics and stability of insect locomotion: a hexapedal model for horizontal plane motions. *Biological cybernetics*, 91(2):76–90, 2004. doi: 10.1007/s00422-004-0498-y.

[73] A Seyfarth, R Blickhan, and JL Van Leeuwen. Optimum take-off techniques and muscle design for long jump. *Journal of Experimental Biology*, 203(4):741–750, 2000.

[74] NE Steenrod. *The topology of fibre bundles*, volume 14. Princeton University Press, 1951.

[75] JR Usherwood, KL Szymanek, and MA Daley. Compass gait mechanics account for top walking speeds in ducks and humans. *Journal of Experimental Biology*, 211(23):3744–3749, 2008. doi: 10.1242/jeb.023416.

[76] Y Wang. *System identification around periodic orbits with application to steady state human walking*. PhD thesis, The Ohio State University, 2013.

[77] PM Wensing and D Orin. 3d-slip steering for high-speed humanoid turns. In *Intelligent Robots and Systems (IROS 2014), 2014 IEEE/RSJ International Conference on*, pages 4008–4013. IEEE, 2014. doi: 10.1109/IROS.2014.6943126.

[78] ER Westervelt, JW Grizzle, and DE Koditschek. Hybrid zero dynamics of planar biped walkers. *Automatic Control, IEEE Transactions on*, 48(1):42–56, 2003. doi: 10.1109/TAC.2002.806653.

[79] S Wiggins. *Normally Hyperbolic Invariant Manifolds in Dynamical Systems*. Springer, New York, New York, 1 edition, 1994. ISBN 9781461243120.