SUPPLEMENTARY MATERIALS: Event–Selected Vector Field Discontinuities Yield Piecewise–Differentiable Flows*

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SM1. F–Derivative formulae for piecewise–differentiable flow.

A piecewise–differentiable function is differentiable almost everywhere [SM8, Theorem 2], and hence its B–derivative at any point is contained in the convex hull of the limit of F–derivatives of its selection functions [SM9, § 4.3]. For completeness and to aid the reader's comprehension, we now derive explicit formulae for F–derivatives of the piecewise–differentiable objects used in Section 4.3 to construct the piecewise–differentiable flow. In general the B–derivative can be obtained via the chain rule [SM9, Theorem 3.1.1].

SM1.1. Budgeted time–to–boundary.

We adopt the notational conventions from Section 4.2.1.

Define $\nu_b^+ : U_b \to \mathbb{R} \cup \{+\infty\}$ using the convention $\min \emptyset = +\infty$ by

$$\forall x \in U_b : \nu_b^+(x) = \min \left\{ \tau_b^{H_j}(x) : b_j < 0 \right\}_{j=1}^{n}, \tag{SM1}$$

then for all $(t, x) \in \mathbb{R} \times U_b$ such that $\nu_b^+(x) \neq t \neq 0$, the forward–time budgeted time–to–boundary τ_b^+ is classically differentiable and

$$D\tau_b^+(t,x) = \begin{cases} \left[0,\ 0_d^\top\right], & t < 0; \\ \left[1,\ 0_d^\top\right], & 0 < t < \nu_b^+(x); \\ \left[0, D\tau_b^H(x)\right], & \nu_b^+(x) < t; \end{cases} \tag{SM2}$$

where in the third case $H \in \{H_j\}_{j=1}^n$ is such that $\tau_b^H(x) = \nu_b^+(x) > 0$.

Define $\nu_b^- : U_b \to \mathbb{R} \cup \{-\infty\}$ using the convention $\max \emptyset = -\infty$ by

$$\forall x \in U_b : \nu_b^-(x) = \min \left\{ \tau_b^{H_j}(x) : b_j > 0 \right\}_{j=1}^{n}, \tag{SM3}$$

*This research was supported in part by: an NSF Graduate Research Fellowship to S. A. Burden; ARO Young Investigator Award #61770 and W911NF–14–1–0573 to S. Revzen; ARL Cooperative Agreements W911NF–08–2–0004 and W911NF–10–2–0016; and National Science Foundation Award #1028237. The views and conclusions contained in this document are those of the authors and should not be interpreted as representing the official policies, either expressed or implied, of the Army Research Laboratory or the U.S. Government. The U.S. Government is authorized to reproduce and distribute for Government purposes notwithstanding any copyright notation herein.

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then for all $(t, x) \in \mathbb{R} \times U_b$ such that $\nu_b^-(x) \neq t \neq 0$, the backward–time budgeted time–to–boundary τ_b^- is classically differentiable and

$$D\tau_b^-(t, x) = \begin{cases} \left[0,\ 0_d^\top\right], & t > 0; \\ \left[1,\ 0_d^\top\right], & \nu_b^-(x) < t < 0; \\ \left[0, D\tau_b^H(x)\right], & t < \nu_b^-(x); \end{cases} \tag{SM4}$$

where in the third case $H \in \{H_j\}_{j=1}^n$ is such that $\tau_b^H(x) = \nu_b^-(x) < 0$.

SM1.2. Budgeted flow–to–boundary. We adopt the notational conventions from Section 4.2.2.

Define $\nu_b^+ : U_b \to \mathbb{R}$ as in (SM1) then for all $(t, x) \in \mathbb{R} \times U_b$ such that $\nu_b^+(x) \neq t \neq 0$, the forward–time flow–to–boundary ζ_b^+ is classically differentiable and

$$D\zeta_b^+(t, x) = \begin{cases} \left[0_d,\ 0_{d\times d}\right], & t < 0; \\ \left[F_b(\phi_b(t, x)),\ D_x\phi_b(t, x)\right], & 0 < t < \nu_b^+(x); \\ \left[0_d, \Upsilon(t, x)\right], & \nu_b^+(x) < t; \end{cases} \tag{SM5}$$

where in the third case $\Upsilon(t, x) = F_b(\phi_b(\tau_b^+(t, x), x))D\tau_b^H(x) + D_x\phi_b(\tau_b^+(t, x), x)$ and $H \in \{H_j\}_{j=1}^n$ is such that $\tau_b^H(x) = \nu_b^+(x)$.

Define $\nu_b^- : U_b \to \mathbb{R}$ as in (SM3) then for all $(t, x) \in \mathbb{R} \times U_b$ such that $\nu_b^-(x) \neq t \neq 0$, the backward–time flow–to–boundary ζ_b^- is classically differentiable and

$$D\zeta_b^-(t, x) = \begin{cases} \left[0_d,\ 0_{d\times d}\right], & t > 0; \\ \left[F_b(\phi_b(t, x)\right],\ D_x\phi_b(t, x)), & \nu_b^-(x) < t < 0; \\ \left[0_d, \Upsilon(t, x)\right], & t < \nu_b^-(x); \end{cases} \tag{SM6}$$

where in the third case $\Upsilon(t, x) = F_b(\phi_b(\tau_b^-(t, x), x))D\tau_b^H(x) + D_x\phi_b(\tau_b^-(t, x), x)$ and $H \in \{H_j\}_{j=1}^n$ is such that $\tau_b^H(x) = \nu_b^+(x)$.

SM1.3. Composite of budgeted time–to– and flow–to–boundary. We adopt the notational conventions from Section 4.2.3.

Combine (20) and (26) to obtain the derivative of φ_b^+ for all $(t, x) \in \mathbb{R} \times U_b$ such that $\nu_b^+(x) \neq t \neq 0$:

$$D\varphi_b^+(t, x) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}, & t < 0; \\ \begin{bmatrix} 0 & 0 \\ F_b(\phi_b(t, x)) & D_x\phi_b(t, x) \end{bmatrix}, & 0 < t < \nu_b^+(x); \\ \begin{bmatrix} 1 & -D\tau_b^H(x) \\ 0 & \Upsilon(t, x) \end{bmatrix}, & \nu_b^+(x) < t; \end{cases} \tag{SM7}$$

where in the third case $\Upsilon(t, x) = F_b(\phi_b(\tau_b^+(t, x), x))D\tau_b^H(x) + D_x\phi_b(\tau_b^+(t, x), x)$ and $H \in \{H_j\}_{j=1}^n$ is such that $\tau_b^H(x) = \nu_b^+(x)$.

Combine (SM4) and (SM6) to obtain the derivative of φ_b^- for all $(t,x) \in \mathbb{R} \times U_b$ such that $\nu_b^-(x) \neq t \neq 0$:

$$D\varphi_b^-(t,x) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}, & t > 0; \\ \begin{bmatrix} 0 & 0 \\ F_b(\phi_b(t,x)) & D_x\phi_b(t,x) \end{bmatrix}, & \nu_b^-(x) < t < 0; \\ \begin{bmatrix} 1 & -D\tau_b^H(x) \\ 0 & \Upsilon(t,x) \end{bmatrix}, & t < \nu_b^-(x); \end{cases} \tag{SM8}$$

where in the third case $\Upsilon(t,x) = F_b(\phi_b(\tau_b^-(t,x),x))D\tau_b^H(x) + D_x\phi_b(\tau_b^-(t,x),x)$ and $H \in \{H_j\}_{j=1}^n$ is such that $\tau_b^H(x) = \nu_b^+(x)$.

SM2. Periodic orbits and B–derivative formulae for phase oscillators. In the following subsections we provide detailed derivations that were deemed too laborious to include in the main text.

SM2.1. Synchronization of first–order phase oscillators. We adopt the notational conventions of Section 8.1, and derive several useful properties of closed–loop dynamics obtained by applying the piecewise–constant feedback in (95) to the system in (81).

First we argue that the synchronized set (82) is a periodic orbit for the closed–loop dynamics. Since (81) consists of d identical subsystems and the feedback in (95) encounters discontinuities synchronously (i.e. all coordinates of the vector field change discontinuously at the same time) at points $\{-\Delta\mathbb{1}, 0_d, +\Delta\mathbb{1}\}$ and at every other point in time the vector field coordinates are identical, we conclude that the trajectory initialized at 0_d remains synchronized for all time.

We now explicitly derive the B–derivative in (86). Fixing a word $\omega : \{1,\ldots,d\} \to B_d$ with corresponding sequence of surfaces crossed $\eta : \{1,\ldots,d\} \to \{1,\ldots,d\}$, we know from Section 7.1 that for all $\varepsilon > 0$ the F–derivative of the selection function ϕ_ω at $(2\varepsilon, \phi(-\varepsilon, 0_d))$ is given by (65),

$$D\phi_\omega(2\varepsilon, \phi(-\varepsilon,0_d)) = D\phi(\varepsilon, 0_d)\left[\prod_{j=1}^{d} D\varphi_{\omega(j)}^+(0,0_d)\right]\begin{bmatrix} 0 \\ D\phi(\varepsilon, \phi(-\varepsilon,0_d)) \end{bmatrix}. \tag{SM9}$$

Here, $D\phi(\varepsilon,0_d)$, $D\phi(\varepsilon,\phi(-\varepsilon,0_d))$ are obtained as in (59) by solving the classical variational equation since $F : \mathbb{R}^d \to T\mathbb{R}^d$ is smoothly extendable to a neighborhood of those segments of the trajectory; noting that for all $s \in [-\varepsilon, 0)$ we have $F(\phi(s,0_d)) = F_{-\mathbb{1}}(0_d)$, $D_xF(\phi(s,0_d)) = 0$ and for all $s \in (0,+\varepsilon]$ we have $F(\phi(s,0_d)) = F_{+\mathbb{1}}(0_d)$, $D_xF(\phi(s,0_d)) = 0$, we conclude that

$$D\phi(s,0_d) = \begin{bmatrix} F_{+\mathbb{1}}(0_d) & I_d \end{bmatrix},\ D\phi(s,\phi(-s,0_d)) = \begin{bmatrix} F_{-\mathbb{1}}(0_d) & I_d \end{bmatrix}. \tag{SM10}$$

For each $j \in \{1,\ldots,d\}$ the derivative $D\varphi_{\omega(j)}^+(0,0_d)$ is given by the matrix in the third case in (29) with the simplifications $\tau_{\omega(j)}^+(0,0_d) = 0$, $\phi_{\omega(j)}(0,0_d) = 0_d$; setting $f_j = F_{\omega(j)}(0_d)$, $g_j^\top = Dh_{\eta(j)(0_d)}$ for each $j \in \{1,\ldots,d\}$, by (66) we have

$$D\varphi_{\omega(j)}^+(0,0_d) = I_{d+1} + \frac{1}{g_j^\top f_j}\begin{bmatrix} 1 \\ -f_j \end{bmatrix}\begin{bmatrix} 0 & g_j^\top \end{bmatrix}. \tag{SM11}$$

Now for all $j \in \{1, \ldots, d\}$ we have $f_j = F_{\omega(j)}(0_d) = \nu \mathbb{1} - \delta\omega(j) \in \mathbb{R}^d$ and $g_j^\top = Dh_{\eta(j)}(0_d) = e_{\eta(j)}^\top \in \mathbb{R}^{1\times d}$, hence $g_j^\top f_j = \nu + \delta$. Since for all $i \in \{1, \ldots, d\}$ with $i > j$ the vector $\omega(i)$ is lexicographically greater than $\omega(j)$, we also have $g_i^\top f_j = \nu + \delta$. These simplifications yield for all $j \in \{1, \ldots, d-1\}$

$$\begin{aligned} &D\varphi^+_{\omega(j+1)}(0, 0_d) D\varphi^+_{\omega(j)}(0, 0_d) \\ &= \left[I_{d+1} + \frac{1}{\nu+\delta} \begin{bmatrix} 1 \\ -f_{j+1} \end{bmatrix} \begin{bmatrix} 0 & e^\top_{\eta(j+1)} \end{bmatrix} \right] \left[I_{d+1} + \frac{1}{\nu+\delta} \begin{bmatrix} 1 \\ -f_j \end{bmatrix} \begin{bmatrix} 0 & e^\top_{\eta(j)} \end{bmatrix} \right] \\ &= I_{d+1} + \frac{1}{\nu+\delta} \begin{bmatrix} 1 \\ -f_{j+1} \end{bmatrix} \begin{bmatrix} 0 & e^\top_{\eta(j+1)} \end{bmatrix} + \frac{1}{\nu+\delta} \begin{bmatrix} 0 \\ f_{j+1} - f_j \end{bmatrix} \begin{bmatrix} 0 & e^\top_{\eta(j)} \end{bmatrix}. \end{aligned} \tag{SM12}$$

Noting that $f_{j+1} - f_j = -2\delta e_{\eta(j)}$, we conclude that $e^\top_{\eta(i)}(f_{j+1} - f_j) = 0$ for all $i \in \{j+1, \ldots, d\}$. This implies that

$$\begin{aligned} \Xi_\omega &= \prod_{j=1}^d D\varphi^+_{\omega(j)}(0, 0_d) \\ &= I_{d+1} + \frac{1}{\nu+\delta} \begin{bmatrix} 1 \\ -f_d \end{bmatrix} \begin{bmatrix} 0 & e^\top_{\eta(d)} \end{bmatrix} + \frac{1}{\nu+\delta} \sum_{j=1}^{d-1} \begin{bmatrix} 0 \\ -2\delta e_{\eta(j)} \end{bmatrix} \begin{bmatrix} 0 & e^\top_{\eta(j)} \end{bmatrix}. \end{aligned} \tag{SM13}$$

Noting for any $f \in \mathbb{R}^d$ with $0_d = 0 \cdot \mathbb{1}_d \in \mathbb{R}^d$ that

$$\begin{bmatrix} 0 & 0_d^\top \\ f & I_d \end{bmatrix} = I_{d+1} + \begin{bmatrix} -1 \\ f \end{bmatrix} \begin{bmatrix} 1 & 0_d^\top \end{bmatrix}, \tag{SM14}$$

defining $f^+ = F_{+\mathbb{1}}(0_d)$ and noting that $f^+ - f_d = -2\delta e_{\eta(d)}$ we have

$$\begin{aligned} \begin{bmatrix} 0 & 0_d^\top \\ f^+ & I_d \end{bmatrix} \prod_{j=1}^d D\varphi^+_{\omega(j)}(0, 0_d) &= I_{d+1} + \begin{bmatrix} -1 \\ f^+ \end{bmatrix} \begin{bmatrix} 1 & 0_d^\top \end{bmatrix} \\ &\quad + \frac{1}{\nu+\delta} \begin{bmatrix} 0 \\ f^+ - f_d \end{bmatrix} \begin{bmatrix} 0 & e^\top_{\eta(d)} \end{bmatrix} \\ &\quad + \frac{1}{\nu+\delta} \sum_{j=1}^{d-1} \begin{bmatrix} 0 \\ -2\delta e_{\eta(j)} \end{bmatrix} \begin{bmatrix} 0 & e^\top_{\eta(j)} \end{bmatrix} \\ &= I_{d+1} - \frac{2\delta}{\nu+\delta} \begin{bmatrix} 0 & 0_d^\top \\ 0_d & I_d \end{bmatrix} + \begin{bmatrix} -1 \\ f^+ \end{bmatrix} \begin{bmatrix} 1 & 0_d^\top \end{bmatrix}. \end{aligned} \tag{SM15}$$

Finally, defining $f^- = F_{-\mathbb{1}}(0_d)$ we have

$$\begin{aligned} D\phi_\omega(0, 0_d) &= \begin{bmatrix} 0 & 0_d^\top \\ f^+ & I_d \end{bmatrix} \prod_{j=1}^d D\varphi^+_{\omega(j)}(0, 0_d) \begin{bmatrix} 0 & 0_d^\top \\ f^- & I_d \end{bmatrix} \\ &= I_{d+1} - \frac{2\delta}{\nu+\delta} \begin{bmatrix} 0 & 0_d^\top \\ 0_d & I_d \end{bmatrix} + \begin{bmatrix} -1 \\ \left(\frac{\nu-\delta}{\nu+\delta}\right) f^- \end{bmatrix} \begin{bmatrix} 1 & 0_d^\top \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{\nu-\delta}{\nu+\delta}\right) f^- & \frac{\nu-\delta}{\nu+\delta} I_d \end{bmatrix}. \end{aligned} \tag{SM16}$$

Restricting to the derivative with respect to state, we find for all words $\omega \in \Omega$ that

$$D_x\phi_\omega(0, 0_d) = \frac{\nu - \delta}{\nu + \delta} I_d. \tag{SM17}$$

Thus the piecewise–differentiable flow ϕ is C^1 with respect to state at $(0, 0_d)$, and (86) follows. Restricting instead to the derivative with respect to time, we find as expected that

$$D_t\phi_\omega(0, 0_d) = \frac{\nu - \delta}{\nu + \delta} f^- = f^+ = F_{+\mathbb{1}}(0_d). \tag{SM18}$$

SM2.2. Synchronization of second–order phase oscillators. We adopt the notational conventions of Section 8.2, and derive several useful properties of closed–loop dynamics obtained by applying the piecewise–constant feedback in (107) to the system in (97).

First we argue that, for any $\beta, \Delta > 0$ there exists $\nu_\beta \in \left(\frac{\alpha}{\beta}, \frac{\alpha+\delta}{\beta}\right)$ such that the trajectory initialized at $(0, \nu\mathbb{1})$ is a periodic orbit for the closed–loop dynamics. Since (97) consists of d identical subsystems and the feedback in (107) encounters discontinuities synchronously (i.e. all coordinates of the vector field change discontinuously at the same time) at points of the form $(\theta\mathbb{1}, \nu\mathbb{1})$ where $\theta \in \{-\Delta, 0, +\Delta\}$ and $\nu > 0$ and at every other point in time the vector field coordinates are identical, we conclude that a trajectory initialized at $(0, \nu\mathbb{1})$ remains synchronized for all time, so the asymptotic behavior of this trajectory can be studied by restricting our attention to the scalar case (i.e. $d = 1$), wherein the dynamics take the simple form

$$d = 1 \implies \ddot{q} = \begin{cases} \alpha - \delta - \beta\dot{q}, \ q \in [-\Delta, 0); \\ \alpha + \delta - \beta\dot{q}, \ q \in [0, +\Delta]; \\ \alpha - \beta\dot{q}, \ \text{else}; \end{cases} \tag{SM19}$$

here we adopt the abuse of notation that $q \in [-\Delta, 0)$ if there exists $x \in [-\Delta, 0) \subset \mathbb{R}$ such that $\pi(x) = q$, and similarly for $q \in [0, +\Delta]$. Clearly if the initial velocity $\dot{q}(0) > 0$ then $\dot{q}(t) > 0$ for all $t > 0$ since $0 < \delta < \alpha$. This implies that $q(t)$ crosses the thresholds $\theta \in \{-\Delta, 0, +\Delta\}$ in sequence. The impact map $P_\beta^{(\theta_1,\theta_2)} : (0, \infty) \to (0, \infty)$ obtained by integrating the flow of (SM19) between any sequential pair of event surfaces $(\theta_1, \theta_2) \in \{(-\Delta, 0), (0, +\Delta), (+\Delta, -\Delta)\}$ is a contraction over velocities with a Lipschitz constant that decreases exponentially with increasing β. Thus the composition $P_\beta = P_\beta^{(0,+\Delta)} \circ P_\beta^{(-\Delta,0)} \circ P_\beta^{(+\Delta,-\Delta)}$ is a contraction for all β sufficiently large. Since furthermore for all β sufficiently large the compact set $\left[\frac{\alpha}{\beta}, \frac{\alpha+\delta}{\beta}\right]$ is mapped to itself under P_β, the Banach contraction mapping principle [SM1] [SM7, Lemma C.35] implies there exists $\nu_\beta \in \left(\frac{\alpha}{\beta}, \frac{\alpha+\delta}{\beta}\right)$ such that $P_\beta(\nu_\beta) = \nu_\beta$. In other words, the trajectory initialized at $(0, \nu_\beta)$ lies on a periodic orbit for (SM19), and hence $(0, \nu_\beta\mathbb{1})$ lies on a periodic orbit for the closed–loop dynamics obtained by applying the feedback in (107) to the system in (97). It is straightforward to verify in this scalar system that solving the variational equation as in Section 7.1 yields

$$\begin{bmatrix} p(s) \\ \dot{p}(s) \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{\beta}\left(1 - e^{-\beta s}\right) \\ 0 & e^{-\beta s} \end{bmatrix} \begin{bmatrix} p(0^+) \\ \dot{p}(0^+) \end{bmatrix} =: X(s) \begin{bmatrix} p(0^+) \\ \dot{p}(0^+) \end{bmatrix}. \tag{SM20}$$

Since the saltation updates are synchronous along the periodic orbit, (103) follows.

We now explicitly derive the B–derivative in (102). Fixing a word $\omega : \{1, \ldots, d\} \to B_d$ with corresponding sequence of surfaces crossed $\eta : \{1, \ldots, d\} \to \{1, \ldots, d\}$, we know from Section 7.1 that for all $\varepsilon > 0$ the F–derivative of the selection function ϕ_ω at $(2\varepsilon, \phi(-\varepsilon, (0, \nu\mathbb{1})))$ is given by (65),

$$\begin{aligned} &D\phi_\omega(2\varepsilon, \phi(-\varepsilon, (0, \nu\mathbb{1}))) \\ &\quad = D\phi(\varepsilon, (0, \nu\mathbb{1})) \left[\prod_{j=1}^{d} D\varphi^+_{\omega(j)}(0, (0, \nu\mathbb{1})) \right] \begin{bmatrix} 0 \\ D\phi(\varepsilon, \phi(-\varepsilon, (0, \nu\mathbb{1}))) \end{bmatrix}. \end{aligned}$$

Here, $D\phi(\varepsilon, (0, \nu\mathbb{1}))$, $D\phi(\varepsilon, \phi(-\varepsilon, (0, \nu\mathbb{1})))$ are obtained as in (59) by solving the classical variational equation since $F : \mathbb{R}^{2d} \to T\mathbb{R}^{2d}$ is smoothly extendable to a neighborhood of those segments of the trajectory; we conclude that

$$\text{(SM21)} \qquad \begin{aligned} \lim_{s \to 0^+} D\phi(s, (0, \nu\mathbb{1})) &= \begin{bmatrix} F_{+\mathbb{1}}(0, \nu\mathbb{1}) & I_d \end{bmatrix}, \\ \lim_{s \to 0^+} D\phi(s, \phi(-s, (0, \nu\mathbb{1}))) &= \begin{bmatrix} F_{-\mathbb{1}}(0, \nu\mathbb{1}) & I_d \end{bmatrix}. \end{aligned}$$

For each $j \in \{1, \ldots, d\}$ the derivative $D\varphi^+_{\omega(j)}(0, (0, \nu\mathbb{1}))$ is given by the matrix in the third case in (29) with the simplifications $\tau^+_{\omega(j)}(0, (0, \nu\mathbb{1})) = 0$, $\phi_{\omega(j)}(0, (0, \nu\mathbb{1})) = (0, \nu\mathbb{1})$; setting $f_j = F_{\omega(j)}(0, \nu\mathbb{1})$, $g_j^\top = Dh_{\eta(j)(0,\nu\mathbb{1})}$ for each $j \in \{1, \ldots, d\}$, by (66) we have

$$\text{(SM22)} \qquad D\varphi^+_{\omega(j)}(0, (0, \nu\mathbb{1})) = I_{d+1} + \frac{1}{g_j^\top f_j} \begin{bmatrix} 1 \\ -f_j \end{bmatrix} \begin{bmatrix} 0 & g_j^\top \end{bmatrix}.$$

For convenience, we let

$$\text{(SM23)} \qquad I_{2d} = \begin{bmatrix} e_1 & \cdots & e_d & \dot{e}_1 & \cdots & \dot{e}_d \end{bmatrix},$$

i.e. for all $j \in \{1, \ldots, d\}$ we let $e_j \in \mathbb{R}^{2d}$ denote the j–th standard Euclidean basis vector and let $\dot{e}_j = e_{d+j} \in \mathbb{R}^{2d}$ denote the $(d + j)$–th such vector; though a mild abuse of the "dot" ("$\dot{}$") notation, this convention simplifies the subsequent exposition. Now for all $j \in \{1, \ldots, d\}$ we have

$$\text{(SM24)} \qquad \begin{aligned} f_j &= F_{\omega(j)}(0, \nu\mathbb{1}) = \begin{bmatrix} \nu\mathbb{1} \\ \alpha\mathbb{1} - \beta\nu\mathbb{1} - \delta\omega(j) \end{bmatrix} \in \mathbb{R}^{2d}, \\ g_j^\top &= Dh_{\eta(j)}(0, \nu\mathbb{1}) = e^\top_{\eta(j)} \in \mathbb{R}^{1 \times 2d}, \end{aligned}$$

and hence for any $i \in \{1, \ldots, d\}$ we have $g_i^\top f_j = \nu$. These simplifications yield for all $j \in \{1, \ldots, d-1\}$

$$\text{(SM25)} \qquad \begin{aligned} &D\varphi^+_{\omega(j+1)}(0, (0, \nu\mathbb{1})) D\varphi^+_{\omega(j)}(0, (0, \nu\mathbb{1})) \\ &\quad = \left[I_{2d+1} + \frac{1}{\nu} \begin{bmatrix} 1 \\ -f_{j+1} \end{bmatrix} \begin{bmatrix} 0 & g_{j+1}^\top \end{bmatrix} \right] \left[I_{2d+1} + \frac{1}{\nu} \begin{bmatrix} 1 \\ -f_j \end{bmatrix} \begin{bmatrix} 0 & g_j^\top \end{bmatrix} \right] \\ &\quad = I_{2d+1} + \frac{1}{\nu} \begin{bmatrix} 1 \\ -f_{j+1} \end{bmatrix} \begin{bmatrix} 0 & g_{j+1}^\top \end{bmatrix} + \frac{1}{\nu} \begin{bmatrix} 0 \\ f_{j+1} - f_j \end{bmatrix} \begin{bmatrix} 0 & g_j^\top \end{bmatrix}. \end{aligned}$$

Noting that $f_{j+1}-f_j = -2\delta\dot{e}_{\eta(j)}$, we conclude that $e^\top_{\eta(i)}(f_{j+1}-f_j) = 0$ for all $i \in \{j+1,\ldots,d\}$. This implies that

$$\begin{aligned}\Xi_\omega &= \prod_{j=1}^{d} D\varphi^+_{\omega(j)}(0,(0,\nu\mathbb{1})) \\ &= I_{2d+1} + \frac{1}{\nu}\begin{bmatrix} 1 \\ -f_d \end{bmatrix}\begin{bmatrix} 0 & e^\top_{\eta(d)} \end{bmatrix} - \frac{2\delta}{\nu}\sum_{j=1}^{d-1}\begin{bmatrix} 0 \\ \dot{e}_{\eta(j)} \end{bmatrix}\begin{bmatrix} 0 & e^\top_{\eta(j)} \end{bmatrix}.\end{aligned} \tag{SM26}$$

Noting for any $f \in \mathbb{R}^{2d}$ with $0_{2d} = 0 \cdot \mathbb{1}_{2d} \in \mathbb{R}^{2d}$ that

$$\begin{bmatrix} 0 & 0^\top_{2d} \\ f & I_{2d} \end{bmatrix} = I_{2d+1} + \begin{bmatrix} -1 \\ f \end{bmatrix}\begin{bmatrix} 1 & 0^\top_{2d} \end{bmatrix}, \tag{SM27}$$

defining $f^+ = F_{+\mathbb{1}}(0,\nu\mathbb{1})$ and noting that $f^+ - f_d = -2\delta\dot{e}_{\eta(d)}$ we have

$$\begin{aligned}&\begin{bmatrix} 0 & 0^\top_{2d} \\ f^+ & I_{2d} \end{bmatrix}\prod_{j=1}^{d} D\varphi^+_{\omega(j)}(0,(0,\nu\mathbb{1})) \\ &\quad = I_{2d+1} + \begin{bmatrix} -1 \\ f^+ \end{bmatrix}\begin{bmatrix} 1 & 0^\top_{2d} \end{bmatrix} + \frac{1}{\nu}\begin{bmatrix} 0 \\ f^+ - f_d \end{bmatrix}\begin{bmatrix} 0 & e^\top_{\eta(d)} \end{bmatrix} \\ &\qquad - \frac{2\delta}{\nu}\sum_{j=1}^{d-1}\begin{bmatrix} 0 \\ \dot{e}_{\eta(j)} \end{bmatrix}\begin{bmatrix} 0 & e^\top_{\eta(j)} \end{bmatrix} \\ &\quad = I_{2d+1} - \frac{2\delta}{\nu}\sum_{j=1}^{d}\begin{bmatrix} 0 \\ \dot{e}_{\eta(j)} \end{bmatrix}\begin{bmatrix} 0 & e^\top_{\eta(j)} \end{bmatrix} + \begin{bmatrix} -1 \\ f^+ \end{bmatrix}\begin{bmatrix} 1 & 0^\top_{2d} \end{bmatrix}.\end{aligned} \tag{SM28}$$

Finally, defining $f^- = F_{-\mathbb{1}}(0,\nu\mathbb{1})$ we have

$$\begin{aligned}&D\phi_\omega(0,(0,\nu\mathbb{1})) \\ &\quad = \begin{bmatrix} 0 & 0^\top_{2d} \\ f^+ & I_{2d} \end{bmatrix}\prod_{j=1}^{d} D\varphi^+_{\omega(j)}(0,(0,\nu\mathbb{1}))\begin{bmatrix} 0 & 0^\top_{2d} \\ f^- & I_{2d} \end{bmatrix} \\ &\quad = I_{2d+1} - \frac{2\delta}{\nu}\begin{bmatrix} 0 & 0^\top_d & 0^\top_d \\ 0_d & 0 & 0 \\ 0_d & I_d & 0 \end{bmatrix} + \left[\begin{bmatrix} -1 \\ f^- \end{bmatrix} - \begin{bmatrix} 0 \\ 0_d \\ 2\delta\mathbb{1} \end{bmatrix}\right]\begin{bmatrix} 1 & 0^\top_d & 0^\top_d \end{bmatrix}.\end{aligned} \tag{SM29}$$

Restricting to the derivative with respect to state, we find for all words $\omega \in \Omega$ that

$$D_x\phi_\omega(0,(0,\nu\mathbb{1})) = \begin{bmatrix} I_d & 0 \\ -\frac{2\delta}{\nu}I_d & I_d \end{bmatrix}. \tag{SM30}$$

Thus the piecewise–differentiable flow ϕ is C^1 with respect to state at $(0,(0,\nu\mathbb{1}))$, and (102) follows. Restricting instead to the derivative with respect to time, we find as expected that

$$D_t\phi_\omega(0,(0,\nu\mathbb{1})) = f^- - \begin{bmatrix} 0_d \\ 2\delta\mathbb{1} \end{bmatrix} = f^+ = F_{+\mathbb{1}}(0,\nu\mathbb{1}). \tag{SM31}$$

SM3. Global piecewise–differentiable flow.

Lemma SM1 (Translation Lemma). *Let $D \subset \mathbb{R}^d$ be open, $F \in EC^r(D)$, $J \subset \mathbb{R}$ be an interval, and $\xi : J \to D$ an integral curve for F. For any $b \in \mathbb{R}$, the curve $\widehat{\xi} : \widehat{J} \to D$ defined by $\widehat{\xi}(t) = \xi(t+b)$ is also an integral curve for F, where $\widehat{J} = \{t : t+b \in J\}$.*

Proof. Clearly $\widehat{\xi} \in PC^r(\widehat{J}, D)$, whence the fundamental theorem of calculus [SM9, Proposition 3.1.1] in conjunction with Lemma 3 implies $\widehat{\xi}$ is an integral curve for F. ∎

Theorem SM2 (Fundamental Theorem on Flows). *If $F \in EC^r(D)$, then there exists a unique maximal flow $\phi \in PC^r(\mathcal{F}, D)$ for F. This flow has the following properties:*

(a) For each $x \in D$, the curve $\phi^x : \mathcal{F}^x \to D$ is the unique maximal integral curve of F starting at x.

(b) If $s \in \mathcal{F}^x$, then $\mathcal{F}^{\phi(s,x)} = \mathcal{F}^x - s = \{t - s : t \in \mathcal{F}^x\}$.

(c) For each $t \in \mathbb{R}$, the set $D_t = \{x \in D : (t,x) \in \mathcal{F}\}$ is open in D and $\phi_t : D_t \to D_{-t}$ is a piecewise–C^r homeomorphism with inverse ϕ_{-t}.

Proof. This proof is a straightforward adaptation of the proof of Theorem 9.12 in [SM7].

Theorem 4 (local flow) shows that there exists an integral curve for F starting at each point $x \in D$. Suppose $\xi, \widetilde{\xi} : J \to D$ are two integral curves for F defined on the same open interval J such that $\xi(t_0) = \widetilde{\xi}(t_0)$ for some $t_0 \in J$. Let $S = \left\{s \in J : \xi(s) = \widetilde{\xi}(s)\right\}$. Clearly $S \neq \emptyset$ since $t_0 \in S$, and S is closed in J by continuity of integral curves. On the other hand, suppose $t_1 \in S$. Applying Theorem 4 (local flow) near $x = \xi(t_1)$, we see that there exists an interval $t_1 \in I \subset \mathbb{R}$ such that $\xi|_I = \widetilde{\xi}|_I$. This implies S is open in J. Since J is connected, $S = J$, which implies $\xi|_J = \widetilde{\xi}|_J$. Thus any two integral curves that agree at one point agree on their common domain.

For each $x \in D$, let $\mathcal{F}^x$ be the union of all domains of integral curves for F originating at x at time 0. Define $\phi^x : \mathcal{F}^x \to D$ by letting $\phi^x(t) = \xi(t)$, where ξ is any integral curve starting at x and defined on an open interval containing 0 and t. Since all such integral curves agree at t by the argument above, ϕ^x is well–defined, and is obviously the unique maximal integral curve starting at p.

Now let $\mathcal{F} = \{(t,x) \in \mathbb{R} \times D : t \in \mathcal{F}^x\}$ and define $\phi : \mathcal{F} \to D$ by $\phi(t,x) = \phi^x(t)$. We also write $\phi_t(x) = \phi(t,x)$. By definition, ϕ satisfies property (a) in the statement of the fundamental theorem: for each $x \in D$, ϕ^x is the unique maximal integral curve for F starting at x. To verify the group laws, fix any $x \in D$ and $s \in \mathcal{F}^x$, and write $y = \phi(s,x) = \phi^x(s)$. The curve $\xi : (\mathcal{F}^x - s) \to D$ defined by $\xi(t) = \phi^x(t+s)$ starts at y, and Lemma SM1 shows that ξ is an integral curve for F. Since ϕ is a function, ξ agrees with ϕ^y on their common domain, which is equivalent to

$$\forall s \in \mathcal{F}^x, t \in \mathcal{F}^{\phi(s,x)} : (s + t \in \mathcal{F}^x) \implies (\phi(t, \phi(s,x)) = \phi(t+s, x)) . \tag{SM32}$$

The fact that $\phi(0,x) = x$ for all $x \in D$ is obvious. By maximality of ϕ^x, the domain of ξ cannot be larger than $\mathcal{F}^y$, which means that $\mathcal{F}^x - s \subset \mathcal{F}^y$. Since $0 \in \mathcal{F}^x$, this implies $-s \in \mathcal{F}^y$, and the group law (SM32) implies that $\phi^y(-s) = x$. Applying the same argument with $(-s, y)$ in place of (s,x), we find that $\mathcal{F}^y + s \subset \mathcal{F}^x$, which is the same as $\mathcal{F}^y \subset \mathcal{F}^x - s$. This proves (b).

Next we show that $\mathcal{F}$ is open in $\mathbb{R} \times D$ (so it is a flow domain) and that $\phi : \mathcal{F} \to D$ is PC^r. Define a subset $W \subset \mathcal{F}$ as the set of all $(t,x) \in \mathcal{F}$ such that ϕ is defined and PC^r on a product neighborhood of (t, x) of the form $J \times U \subset \mathcal{F}$, where $J \subset \mathbb{R}$ is an open interval containing 0 and t and $U \subset D$ is a neighborhood of x. Then W is open in $\mathbb{R} \times D$, and the restriction $\phi|_W \in PC^r(W, D)$, so it suffices to show that $W = \mathcal{F}$. Suppose this is not the case. Then there exists some point $(\tau, x_0) \in \mathcal{F} \setminus W$. For simplicity, assume $\tau > 0$; the argument for $\tau < 0$ is similar (and can be obtained, for instance, by considering the flow for $-F$).

Let $t_0 = \inf\{t \in \mathbb{R} : (t, x_0) \notin W\}$ (see Fig. 9.6 in [SM7]). By Theorem 4 (local flow), ϕ is defined and PC^r in some product neighborhood of $(0, x_0)$, so $t_0 > 0$. Since $t_0 \leq \tau$ and $\mathcal{F}^{x_0}$ is an open interval containing 0 and τ, it follows that $t_0 \in \mathcal{F}^{x_0}$. Let $y_0 = \phi^{x_0}(t_0)$. By Theorem 4 (local flow) again, there exists $\varepsilon > 0$ and a neighborhood U_0 of y_0 such that $(-\varepsilon, \varepsilon) \times U_0 \subset W$. We will use the group law (SM32) to show that ϕ admits a PC^r extension to a neighborhood of (t_0, x_0), which contradicts our choice of t_0.

Choose some $t_1 < t_0$ such that $t_1 + \varepsilon > t_0$ and $\phi^{x_0}(t_1) \in U_0$. Since $t_1 < t_0$, we have $(t_1, x_0) \in W$, so there is a product neighborhood $(t_1 - \delta, t_1 + \delta) \times U_1 \subset W$ for some $\delta > 0$. By definition of W, this implies ϕ is defined and PC^r on $[0, t_1 + \delta) \times U_1$. Because $\phi(t_1, x_0) \in U_0$, we can choose U_1 small enough that ϕ maps $\{t_1\} \times U_1$ into U_0. Define $\widetilde{\phi} : [0, t_1 + \varepsilon) \times U_1 \to D$ by

$$\forall (t, x) \in [0, t_1 + \varepsilon) \times U_1 : \widetilde{\phi}(t, x) = \begin{cases} \phi_t(x), & x \in U_1,\ 0 \leq t < t_1, \\ \phi_{t - t_1} \circ \phi_{t_1}(x), & x \in U_1,\ t_1 - \varepsilon < t < t_1 + \varepsilon. \end{cases}$$

The group law for ϕ guarantees that these definitions agree where they overlap, and our choices of U_1, t_1, and ε ensure that this defines a PC^r map. By Lemma SM1, each map $t \mapsto \widetilde{\phi}(t, p)$ is an integral curve of F, so $\widetilde{\phi}$ is a PC^r extension of ϕ to a neighborhood of (t_0, x_0), contradicting our choice of t_0. This completes the proof that $W = \mathcal{F}$.

Finally, we prove (c). The fact that D_t is open is an immediate consequence of the fact that $\mathcal{F}$ is open. From part (b) we deduce that

$$\begin{aligned} x \in D_t &\implies t \in \mathcal{F}^x \implies \mathcal{F}^{\phi_t(x)} = \mathcal{F}^x - t \\ &\implies -t \in \mathcal{F}^{\phi_t(x)} \implies \phi_t(x) \in D_{-t}, \end{aligned}$$

which shows that ϕ_t maps D_t to D_{-t}. Moreover, the group laws then show that $\phi_{-t} \circ \phi_t$ is equal to the identity on D_t. Reversing the roles of t and $-t$ shows that $\phi_t \circ \phi_{-t}$ is the identity on D_{-t}, which completes the proof. ■

SM4. Perturbation of differential inclusions. In the proof of the perturbation results of Section 6, we relied on a result due to Filippov. For completeness, we reproduce the statement of the result we required.

Assumption 1 ([SM6, Chapter 2, §8, Theorem 1]). *In the domain $\mathcal{F}$ a set–valued function $F(t, x)$ satisfies the* basic conditions *if for all $(t, x) \in \mathcal{F}$ the set $F(t, x)$ is nonempty, bounded, closed, and convex, and furthermore the function F is upper semicontinuous in t, x.*

Here, $\mathcal{F}$ is understood to be a subset of $\mathbb{R} \times \mathbb{R}^d$, and F is upper semicontinuous as a *multifunction* $F : \mathcal{F} \to 2^{\mathbb{R}^d}$ [SM3, §2.1], i.e. for all $(t, x) \in \mathcal{F}$, $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall (s, y) \in (t, x) + B_\delta(0) : F(s, y) \subset F(t, x) + B_\varepsilon(0) \tag{SM33}$$

where we adopt the usual notation in a Banach space X,

$$\forall x \in X,\ B \subset X : x + B = \{x + y : y \in B\}. \tag{SM34}$$

As in [SM6, Chapter 2, §8], for any $\widetilde{F} : \mathcal{F} \to 2^{\mathbb{R}^d}$ we define the deviation $d_{\mathcal{F}}(\widetilde{F}, F)$ as

$$d_{\mathcal{F}}(\widetilde{F}, F) = \inf\left\{\delta > 0 \mid \forall (t,x) \in \mathcal{F} : \widetilde{F}(t,x) \subset [\operatorname{co} F\,(t + B_\delta(0), x + B_\delta(0))] + B_\delta(0)\right\} \tag{SM35}$$

where for any $U \subset \mathbb{R}^d$ the set $\operatorname{co} U$ denotes the convex hull of points in U.

Theorem SM3 ([SM6, Chapter 2, §8, Theorem 1]). *Let $F(t,x)$ satisfy Assumption 1 (differential inclusion basic conditions) in the open domain $\mathcal{F}$; $t_0 \in [a,b]$, $(t_0, x_0) \in \mathcal{F}$; let all the solutions of the problem*

$$\dot{x} \in F(t,x),\ x(t_0) = x_0 \tag{SM36}$$

exist for all $t \in [a,b]$ and their graphs lie in $\mathcal{F}$.

Then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $\widetilde{t}_0 \in [a,b]$, $\widetilde{x}_0$ and $\widetilde{F}(t,x)$ satisfying the conditions

$$\left|\widetilde{t}_0 - t_0\right| \le \delta,\ \|\widetilde{x}_0 - x_0\| \le \delta,\ d_{\mathcal{F}}(\widetilde{F}, F) \le \delta$$

and Assumption 1 (differential inclusion basic conditions), each solution of the problem

$$\dot{\widetilde{x}} \in \widetilde{F}(t, \widetilde{x}),\ \widetilde{x}(\widetilde{t}_0) = \widetilde{x}_0 \tag{SM37}$$

exists for all $t \in [a,b]$ and differs from some solution of (SM36) by not more than ε.

Here, a “solution of the problem (SM36)” on the interval $[a,b] \subset \mathbb{R}$ is an absolutely continuous function $y : [a,b] \to \mathbb{R}^d$; its “graph lies in $\mathcal{F}$” if $\{(t, y(t)) : t \in [a,b]\} \subset \mathcal{F}$.

SM5. A non–event–selected vector field. To obtain the results above, we restricted the class of vector field discontinuities to be event–selected C^r as in Definition 2. The conditions imposed in our definition are equivalent to [SM4, Equation (7.69)], [SM5, Equation (40)] in the case where the vector field is discontinuous across two smooth surfaces that intersect transversely, but these references do not indicate how to generalize to an arbitrary number of surfaces that are not required to be transverse, which case is accommodated by our definition. Figure SM1 provides an illustration of the main phenomenon precluded by our definition: *branching* [SM2] of the flow.

REFERENCES

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fundamenta Mathematicae, 3 (1922), pp. 133–181.

[2] S. A. Burden, H. Gonzalez, R. Vasudevan, R. Bajcsy, and S. S. Sastry, *Metrization and Simulation of Controlled Hybrid Systems*, IEEE Transactions on Automatic Control, 60 (2015), pp. 2307–2320, http://dx.doi.org/10.1109/TAC.2015.2404231, http://arxiv.org/abs/1302.4402.

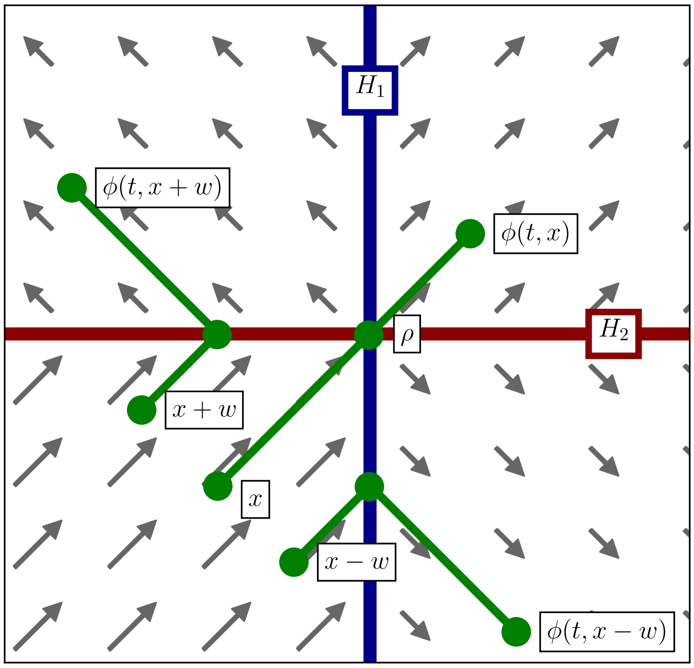

Figure SM1: Illustration of vector field discontinuities precluded by Definition 2. The vector field $F : \mathbb{R}^2 \to T\mathbb{R}^2$ is piecewise–constant, and discontinuous across the standard coordinate hyperplanes $H_1, H_2 \subset \mathbb{R}^2$. The flow $\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ *branches* along the portions of H_1 and H_2 that abut the upper–right quadrant, i.e. ϕ is discontinuous at $(0, \rho) \in \mathbb{R} \times \mathbb{R}^2$. This is clear since initial conditions $x \pm w$ flow to the interior of quadrants that are distinct from that of the trajectory initialized at x.

[3] F. H. Clarke, *Optimization and nonsmooth analysis*, vol. 5, Society for Industrial and Applied Mathematics, 1990.

[4] M. Di Bernardo, C. J. Budd, P. Kowalczyk, and A. R. Champneys, *Piecewise–smooth dynamical systems: theory and applications*, Springer, 2008.

[5] L. Dieci and L. Lopez, *Fundamental matrix solutions of piecewise smooth differential systems*, Mathematics and Computers in Simulation, 81 (2011), pp. 932–953, http://dx.doi.org/10.1016/j.matcom.2010.10.012.

[6] A. F. Filippov, *Differential equations with discontinuous righthand sides*, Springer, 1988.

[7] J. M. Lee, *Introduction to smooth manifolds*, Springer–Verlag, 2012.

[8] R. Rockafellar, *A property of piecewise smooth functions*, Computational Optimization and Applications, 25 (2003), pp. 247–250, http://dx.doi.org/10.1023/A:1022921624832.

[9] S. Scholtes, *Introduction to piecewise differentiable equations*, Springer–Verlag, 2012, http://dx.doi.org/10.1007/978-1-4614-4340-7.