Data driven models of legged locomotion

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Keywords: legged locomotion, oscillator, phase, hybrid system, DDFA, data-driven Floquet analysis

ABSTRACT

Legged locomotion is a challenging regime both for experimental analysis and for robot design. From biology, we know that legged animals can perform spectacular feats which our machines can only surpass on some specially controlled surfaces such as roads. We present a concise review of the theoretical underpinnings of Data Driven Floquet Analysis (DDFA), an approach for empirical modeling of rhythmic dynamical systems. We provide a review of recent and classical results which justify its use in the analysis of legged systems.

1. LOCOMOTION AS AN OSCILLATOR

Locomotion is a process whereby the body moves through space. Up to changes of shape the body is related to its history by a continuous trajectory in the space of body frame position and orientation – the Lie group $\mathbf{SE}(3)$. The mathematical structure representing such a system[1] is that of a "*principal fiber bundle*" – a product-like construction pairing the shape-space of the body $\mathcal{S}$, with $\mathbf{SE}(3)$. Together these are the body's "*configuration space*" $\mathcal{Q}$. When locomoting, bodies exert forces on a medium in their environment to produce the reaction forces that move them. Mechanics dictates a relationship – the "*connection*" – between shape velocity and configuration velocity, i.e. between $\mathsf{T}\mathcal{S}$ and $\mathsf{T}\mathcal{Q}$. This highly abstracted view is sometimes used in the undulatory locomotion literature,[1] where the arbitrariness of body frame choice makes its explicit treatment necessary. However, it remains a valid representation for virtually all self-propelling bodies. In an ideal world, we would be able to predict robust and reliable estimates of the connection for any body of interest, in any medium. Such an estimate would be a complete and self-contained representation of how that body could move through the medium.

Due to the daunting complexity of body-medium interactions most work on locomotion is focused on simplified models. Low dimensional, sprung mass models have captured salient features of the dynamics of many legged locomotion systems.[2] These models can be organized using the "*templates and anchors*" hypotheses (TAH):[3] *"Animals have many DOF, but move 'as if' they have only a few. Animals limit pose to a behaviorally relevant family of postures"*. Template-and-Anchor is a relationship between models of locomotion – one being more elaborate, the other more parsimonious. Both are assumed to be good long-term predictors of the motion, making the template a dimensionally reduced model of the anchor.

At this point most treatments of locomotion in biomechanics and robotics would proceed to discuss the structure of various templates, and the predictions obtained from simulation of high-dimensional anchored models. Such approaches are **model-driven** – the discussion is focused primarily on models, their justification from assumptions about the physics and biology, and the derivaiton of governing parameters for the models. In our **data-driven** approach the focus is on large information-rich datasets. We used first principles to define a broad class of models, sidestepping the issue of assumptions about the fundamental physics or biology, and instead assumed a general mathematical form for the equations of motion. This does not imply that data-driven models can have any arbitrary structure. The system is observed through instrumentation limited by observation noise, and the system itself is subject to physical uncertainties. Consequently, the dynamics of data-driven models must be "*generic*" – they would persist with little change under appropriate classes of mathematical perturbation. In other words, data-driven models must be observable in noisy conditions. As an example, matrices describing data-driven linear models are not expected to exhibit duplicate eigenvalues or nontrivial Jordan blocks: both these properties do not persist under small perturbations of the matrix elements.

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Because they require weaker assumptions, data-driven models are less dependent on the investigator's expectation of the governing physics or biology. Data-driven models are closely tied to the raw data and thus often better predictors than those produced by model-driven approaches. Combined, these features offer an opportunity to refute core assumptions of pervasive models. Additionally, off-line data-driven modeling is closely related to on-line machine learning, and may offer clues towards automated control of the system being modeled.

Animal locomotion at moderate and high speeds often exhibits a rhythmic structure: the shape of the body undergoes changes that are periodic or nearly so. One explanation of this observation is that an underlying periodic solution to the equations of motion persists by virtue of being exponentially attractive (a generic property among stable periodic solutions). The observed body changes can then be considered as trajectories of a deterministic nonlinear system which is an "*oscillator*" – the stability basin of a "(limit) cycle" – that we happened to be observing through noisy instrumentation, and that was furthermore perturbed by the injection of noise from the environment. We have called this class of systems "rhythmic systems[4]" to distinguish them from the familiar and purely deterministic "periodic systems".

Dynamical systems offers many classical results on stability of smooth oscillators,[5–10] as well as modern results on "normal forms" and "invariant manifolds".[11–13] These later results suggest (see §2) that recovery time-constants for some perturbations could be very short and others very long. Perturbed states with long time constants should be observed more readily than those with short time constants. Such time-scale differences could produce the observations stated in the TAH. More recent results regarding discontinuous ("hybrid") oscillators show that template-like dimensionality reduction is a structurally stable feature of their structure, and that this structure persists even when multiple limbs are expected to generate discontinuities at the same time (see §3) .

DDFA was first developed as a means for "finding the template in the data[14]". It has evolved into a collection of methods of far broader utility. At its core is the fact that exponentially stable oscillators are always constant coefficient linear systems, when expressed in their "Floquet Normal Form" (see Theorem 4). In DDFA we aim to find the coordinate change that produces the Floquet Normal Form for the oscillator being observed. The potential payoff is multi-fold: First, the rhythmic shape changes representing the limit cycle in $\mathcal{B} \subseteq \mathcal{S}$ can be accurately estimated, providing a prediction of future motions based on current state. Second, any template-and-anchor relationships would be revealed as slower template modes would correspond to larger eigenvalues in the matrices produced by DDFA[4]). Third, the complexity of the rhythmic shape changes in $\mathcal{B}$ can be removed, instead having that shape dynamic expressed as a Linear Time Invariant (LTI) system.[15] Fourth, the structure of this linear system may be exploitable for improving modeling, control and estimation of the oscillator (see §4).

2. CLASSICAL OSCILLATOR THEORY

In the review of classical results that follows we will only consider properties of $\mathcal{C}^\infty$ vector fields with exponentially stable limit cycles. We will assume that all matrices are generic in the sense of having no duplicate eigenvalues, and these eigenvalues being "*non-resonant*". First we note results for Linear Time-Periodic (LTP) systems, the topic of Floquet's seminal paper.[6] Let $A(\cdot) : \mathbb{R} \to \mathbb{R}^{n \times n}$ be LTP with period T:

$$\forall t \in \mathbb{R} : \ A(t+T) = A(t), \ \dot{x}(t) = A(t)x(t) \tag{1}$$

The unique solution to (1) is given by $x(t) = \Phi(t, t_0)x(t_0)$, where $\Phi(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbf{GL}(n, \mathbb{R})$ is the "*state transition matrix*".[15] The key result that will be used in the analysis of nonlinear oscillators is the "*Real Floquet Representation Theorem*",[6,9,16,17] shown by Floquet himself.

Theorem 1. *[[17] **Theorem 2.3 pp. 254]:** The state transition matrix $\Phi(\cdot, t_0)$ of equation (1) has a decomposition into real "Floquet factors" $F \in \mathbb{R}^{n \times n}$ and $Q(\cdot) : \mathbb{R} \to \mathbf{GL}(n, \mathbb{R})$. $Q(\cdot)$ is $2T$-periodic (and is T-periodic up to sign): $\Phi(t, t_0) = Q(t)e^{(t-t_0)\Lambda}Q^{-1}(t_0)$, where Λ is in Jordan normal form. Furthermore, if $A(\cdot)$ is a $\mathcal{C}^\infty$ function, then so is $Q(\cdot)$.*

Corollary 1. *Using $z := Q^{-1}(t)x$ converts Eqn. (1) to a linear time-invariant (LTI) system $\dot{z} = \Lambda z$*

Consider an open set $E \subset \mathbb{R}^n$ and a $\mathcal{C}^\infty$ vector field $f : E \to \mathbb{R}^n$ defining the nonlinear ordinary differential equation (ODE) $\dot{x} = f(x)$ such that the associated "*flow*"[18] $\phi : \mathbb{R} \times E \to E$ satisfying $\frac{\partial}{\partial t}\phi(t, x) = f(x)$ is $\mathcal{C}^\infty$. As a notational convenience we will often write $\phi_t(x) := \phi(t, x)$ when wishing to emphasize the action of the "*time t map*". Further assume that this system has a single limit cycle $\gamma(\cdot)$ with exponentially stable image Γ, stability basin $\mathcal{B}$ and least period T. We refer to $\mathcal{B}$ and the dynamics within as the "*oscillator*".

Poincaré maps A "*transverse section*" or "*Poincaré section*" S at a point $p \in \Gamma$ is a $\mathcal{C}^\infty$ embedded submanifold of co-dimension 1 that satisfies the "*transversality*" property $f(p) \notin T_pS$, where T_pS is the tangent space to S at p. For points $p \in \Gamma$, $\phi_T(p) = p$ where T is the period of γ. If S is a Poincaré section and $p \in S \cap \Gamma$, it can be shown that there exist relatively open neighborhoods $W_0, W_1 \subset S$ with $p \in W_0 \cap W_1$ and a $\mathcal{C}^\infty$ "*first impact time*" $\tau : W_0 \to \mathbb{R}_{>0}$ such that $\forall x \in W_0$, $\phi(\tau(x_0), x) \in W_1$ and $\forall t \in (0, \tau(x)) : \phi(t, x) \notin S$.[7,8,10] Given a Poincaré section S at a point p in an asymptotically stable orbit Γ, we define a "*Poincaré map*" $\Theta : W_0 \subset S \to S$ by $\Theta(s_0) = \phi(\tau(s_0), s_0)$, W_0 and τ as given above. Following,[7,10] we know:

Theorem 2. *[[7] **Theorem 4.3.21/ Prop. 4.3.23, pp. 272, 274**] Let $\mathbf{D}\Theta_p : \mathsf{T}_pS_p \to \mathsf{T}_pS_p$ denote the differential of the Poincaré map $\Theta_p : W_0 \cap S_p \to S_p$, with $p \in W_0 \cap S_p$. Then for any point $q \in \Gamma$ and any Poincaré section S_q at q, the linear map $\mathbf{D}\Theta_p$ is similar to the linear map $\mathbf{D}\Theta_q$ and hence has the same eigenvalues. Furthermore, $\forall p, q \in \Gamma : \mathrm{eig}(\mathbf{D}\phi_T(p)) = \mathrm{eig}(\mathbf{D}\Theta_q) \cup \{1\}$.*

The eigenvalues of $\mathbf{D}\Theta_p$, $\lambda_j = e^{\mu_j}, j = 1, \ldots, (n-1)$ are called *Floquet (or characteristic) multipliers* of Γ, and the numbers $\mu_j, j = 1, \ldots, (n-1)$ are called *Floquet (or characteristic) exponents* of Γ.[7,8] Note that the characteristic exponents are not uniquely defined. Smoothness implies that the stability of the linearization $\mathbf{D}\Theta_p$ of a Poincaré map governs the stability of Γ in continuous time. For a proof, see.[7–10]

Asymptotic phase We define two points $x, y \in E$ to be "*asymptotically equivalent*" if and only if $\lim_{t\to\infty} \|\phi_t(x) - \phi_t(y)\| = 0$ and write $x \sim y$. This is an equivalence relation and thus partitions E into equivalence classes. No two points in Γ are asymptotically equivalent, but all points in $\mathcal{B}$ are equivalent to some point in Γ; let that representative be $\theta(x) \in \Gamma$. We refer to $\theta(x)$ as the "*asymptotic phase*" of x. Given $p \in \Gamma$, the equivalence class $S_p := \theta^{-1}(\{p\})$ as an "*isochron*", as named by Winfree[19] and Guckenheimer.[20]

Theorem 3. *[[20] **Theorem A/ Appendix A, pp. 261, 266**] $\forall \beta \in \Gamma$: the isochron S_β is a $\mathcal{C}^\infty$ injective immersion of $\mathbb{R}^{n-1}$ and the map $\theta : \mathcal{B} \to \Gamma$ is $\mathcal{C}^\infty$. Each isochron S_β is transverse to the vector field: $\forall x \in S_\beta$, $f(x) \notin T_xS_\beta$. Furthermore, $\mathcal{B} = \bigcup_{\beta\in\Gamma} S_\beta$. Isochrons are permuted by the flow: $S_{\theta(\phi(t,x))} = \phi(t, S_{\theta(x)})$*

2.1 Floquet theory applied to smooth nonlinear oscillators

There is a well-established body of literature on "*normal forms*" for vector fields near invariant manifolds.[11–13,21] These are changes of coordinates under which the behavior of a dynamical system assumes a simple form.

Let $A(t) := \mathbf{D}f(\gamma(t))$, and note that $A(\cdot)$ is T-periodic. Under a genericity condition, there exists a smooth change of coordinates transforming $\dot{x} = f(x)$ into the system $\dot{\beta} = 1/T$, $\dot{z} = A(\beta)z$ [[22] Theorem 3 pp. 798]. Applying the transformation $(\beta, z) \mapsto (\beta, Q^{-1}(\beta)z)$ of Corollary 1 above to this result yields the following theorem specialized to our assumptions of a $\mathcal{C}^\infty$ vector field and exponentially stable limit cycle (note: $Q(\cdot)$ is in fact T-periodic w.r.t. a demiperiodic pseudochart[11]).

Theorem 4. *[[22] **Corollary of Theorem 3, pp. 798**] Assuming a generic condition holds for the Floquet multipliers of Γ, there is a neighborhood U of Γ and a $\mathcal{C}^\infty$ diffeomorphism $P : U \to S^1 \times \mathbb{R}^{n-1}$ such that $P^{-1}(S^1 \times \{0\}) = \Gamma$ and $\forall \beta \in S^1 : P^{-1}(\{\beta\} \times \mathbb{R}^{n-1}) = S_{P^{-1}(\beta,0)}$. In the (demiperiodic) coordinate change $P(x) = (\beta, z)$ the dynamics of the oscillator are:*

$$\dot{\beta} = 1/T \qquad\qquad \dot{z} = \Lambda z \tag{2}$$

Eqn. (2) is the "*Floquet normal form*" for $\dot{x} = f(x)$ in the neighborhood U. Since P is a diffeomorphism, any generalized eigenspace L of Λ in Eqn. (2) can be pulled back to $P^{-1}(L)$, an invariant manifold of $\dot{x} = f(x)$. For a generic system as those arising in DDFA, Λ can be assumed to be in block 2×2 diagonal form. With e_i the standard basis, $f_i : \mathbb{R} \to \mathcal{B}$ given by $f_i(t) := P^{-1}(t/T \bmod 1, e^{t\Lambda}e_i))$ is called a "*Floquet mode*". Letting $x_0 \in U$ and $(\beta_0, z_0) := P(x_0)$, the theorem shows that: $\phi_t(x_0) = P^{-1}\left((t/T + \beta_0) \bmod 1, e^{t\Lambda}z_0\right)$, a sum of Floquet modes in the z coordinates.

3. HYBRID OSCILLATOR THEORY

Most models of legged locomotion are not $\mathcal{C}^\infty$ smooth systems. Instead these models are primarily "*hybrid systems*" – systems with vector fields that contain discontinuities. These discontinuities usually represent the changes in the dynamics that occur when a foot switches between non-contact, sliding (dynamic) friction and static contact with the ground. Informally speaking, hybrid systems can be defined in two ways: as a finite collection of smooth dynamical systems spliced together through "*reset maps*" between their domains, or as a single dynamical system which is smooth except for discontinuities restricted to co-dimension 1 sub-manifolds. An "*execution*" of the hybrid system corresponds to the notion of a trajectory in an ODE. Points along the execution that correspond to discontinuities in the vector field are "*transitions*".

Oscillators with Isolated Transitions When an oscillator can only encounter transitions on isolated locations along the limit cycle, it is convenient to represent its state as a disjoint union of a smooth connected manifolds with boundary $M := \coprod_{k\in J} M_k$ (note: these may have different dimensions) – a "*smooth hybrid manifold*".[23,24] To these naturally correspond their "*hybrid tangent space*" $\mathsf{T}M$ and "*hybrid boundary*" ∂M. As a consequence of the topological structure, a continuous map $R : M \to N$ between smooth hybrid manifolds $M = \coprod_{j\in J} M_j$ and $N := \coprod_{i\in I} N_i$ must map each M_j into a unique N_i, allowing the concept of a smooth map to be naturally extended to smooth hybrid manifolds.

A "*(piecewise defined) hybrid dynamical system*" is a tuple $H = (D, F, G, R)$ where the state D is a smooth hybrid manifold; the vector field is a smoth function $F : D \to \mathsf{T}D$; the "*guard*" $G \subseteq \partial D$ is an open subset of ∂D in the induced topology of ∂D, and the "*reset map*" $R : G \to D$ is a smooth map. Further assume that F is outward pointing on G. An execution in this class of hybrid systems consists of an almost everywhere $\mathcal{C}^1$ path $p : [0, T] \to D$. For any open interval $(s, t) \subseteq [0, T]$ for which p is continuous, $\dot{p}(t) = F(p(t))$. For any point $0 \le t \le T$ for which $p(t^+) \ne p(t^-)$, $p(t^-) \in G$ and $p(t^+) = R(p(t^-))$.

Assume there exists a periodic execution $\gamma : \mathbb{R} \to D$ such that $\gamma(t) = \gamma(t + T)$, and such that transition times have no accumulation points. We have recently shown:

THEOREM 5 ([23,24] THEOREM 1 EXACT REDUCTION). *Let γ be a periodic orbit that undergoes isolated transitions in a hybrid dynamical system $H = (D, F, G, R)$ with $D = \coprod_{j\in J} D_j$, $P : U \to \Sigma$ a Poincaré map for γ, $m = \min_j \dim D_j$, and suppose there exists a neighborhood $V \subset U$ of $\{\xi\} = \gamma \cap U$ and $r \in \mathbb{N}$ such that $\mathrm{rank}\mathbf{D}P^m(x) = r$ for all $x \in V$. Then there exists an $(r+1)$–dimensional hybrid embedded submanifold $M \subset D$ and a hybrid open set $W \subset D$ for which $\gamma \subset M \cap W$ and trajectories starting in W contract to M in finite time.* We conclude that the hybrid structure of legged locomotion dynamics can naturally lead to dimensionality reduction as expressed by the TAH. One may envision a form of control whereby disturbances in the state of the CoM are transferred into the limbs, and then "lost" by having the limbs collide with the substrate, resetting their state (see[23,24] Sec. IV), leading to robust low dimensional asymptotic behaviors.

However, the hybrid structure might only be approximate, e.g. feet might not obtain perfect purchase on the ground, or kinetic energy might not be entirely annihilated by an assumed plastic collision. We have shown that under milder conditions the finite time convergence to M is replaced by a super-exponential convergence rate:

THEOREM 6 ([23,24] THEOREM 2 APPROXIMATE REDUCTION). *Let γ be an exponentially stable periodic orbit undergoing isolated transitions in a hybrid dynamical system $H = (D, F, G, R)$ with $D = \coprod_{j\in J} D_j$, $P : U \to \Sigma$ a Poincaré map for γ with fixed point $\{\xi\} = \gamma \cap \Sigma$. Let $m = \min_j \dim D_j$, and $r = \mathrm{rank}\mathbf{D}P^m(\xi)$. Then there exists an $(r+1)$–dimensional hybrid embedded submanifold $M \subset D$ such that for any $\varepsilon > 0$ there exists a hybrid open set $W^\varepsilon \subset D$ for which $\gamma \subset M \cap W^\varepsilon$ and the Euclidean distance from trajectories starting in W^ε to M contracts by ε each cycle.* Note that unlike the previous theorem, here the result depends only on the rank of $\mathbf{D}P^m$ at point ξ on the limit cycle.

Asymptotics of hybrid oscillators with isolated transitions are the same as those of smooth systems:

THEOREM 7 ([23,24] THEOREM 3 REDUCTION IS SMOOTHABLE). *Let $H = (M, F, G, R)$ be a hybrid dynamical system with $M = \coprod_{j\in J} M_j$. Suppose $\dim M_j = n$ for all $j \in J$, $R(G) \subset \partial M$, $\partial M = G \cup R(G)$, $G \cap R(G) = \emptyset$, R is a hybrid diffeomorphism onto its image, and F is inward–pointing along $R(G)$. Then the topological quotient $\tilde{M} = M/(G \overset{R}{\sim} R(G))$ may be endowed with a smooth structure such that: (1) the projection $\pi : M \to \tilde{M}$ restricted* *to each M_j, $\pi|_{M_j} : M_j \to \tilde{M}$ is a smooth embedding; (2) exists a smooth vector field $\tilde{F} : \tilde{M} \to \mathsf{T}\tilde{M}$ where any execution $x : [0,T] \to M$ of H maps to an integral curve of $\tilde{F}$ via π: $\frac{d}{dt}\pi \circ x(t) = \tilde{F}(\pi \circ x(t))$.* Thus dynamics on M are "*smoothable*": there exists a $\mathcal{C}^0$ coordinate change, $\mathcal{C}^\infty$ on the interior of each M_j, which converts the dynamics to those of a classical smooth oscillator. This implies in turn that the dynamics on M may be rectified into an LTI system in Floquet Normal Form.

Event Selected Hybrid Systems Many animals exhibit gaits where transitions are not isolated. They use "*multiple contact gaits*": several legs touch down or lift off approximately at the same time, e.g. quadruped "trotting" or "pacing".[25,26] Thus several guards intersect on the limit cycle itself. The literature on hybrid systems provides little treatment of oscillators with intersecting or tangent guards, except for one notable exception.[27] In recent work,[28] we have developed a framework for the analysis of a broad class of such systems – "*Event Selected $\mathcal{C}^r$ (Hybrid Systems)*" or ESS hereon. A key insight regarding ESS is that flows of ESS are topologically conjugate to flow-boxes (see Theorem 8). Similar to flow-boxes of smooth systems, this structure in an ESS persists under perturbations of the flow [[28] §5].

Formal definition Assume an open connected domain $D \subseteq \mathbb{R}^n$ containing a point ρ. We define $\mathcal{C}^\infty$ "*event functions*" $h_k : D \to \mathbb{R}$ $k = 1 \ldots m$, such that $h_k(\rho) = 0$. These generate a hybrid system as follows. Let $\eta : \mathbb{R} \to \{-1, 1\}$ map non-negative reals to 1, and negative reals to -1. Let there be a $\mathcal{C}^\infty$ vector field $F_B : B \times D \to \mathsf{T}D$ where $B := \{-1, 1\}^m$ with the discrete topology. Define $F : D \to \mathsf{T}D$ by $F(x) := F_B([\eta \circ h_1(x), \eta \circ h_2(x), \ldots, \eta \circ h_m(x)], x)$ and note that F is smooth everywhere except possibly where some $h_k = 0$. An execution consists of an almost everywhere $\mathcal{C}^1$ path $p : [0,T] \to D$, such that $\dot{p}(t) = F(p(x))$

As defined, this class of systems can exhibit quite complex behaviors. For the study of hybrid oscillators appearing in locomotion, we add one key assumption: $\forall b \in B, x \in D, 1 \leq k \leq m : \nabla h_k(x)\dot{F}(b,x) > 0$. Systems that meet this assumption we consider "*Event Selected Systems (near ρ)*" (ESS). For ESS, each h_k is strictly monotone increasing.

General properties of ESS flows Surprisingly, ESS produce a continuous and piecewise smooth flow – one that is in the class of PC^∞ functions (see[29] ch. 3). Such functions are not differentiable everywhere in the conventional (Frechet) sense, but are Bouligand differentiable (B-differentiable). For ESS, the B-derivative $\mathbf{D}f(x;\delta)$ of the flow is continuous and piecewise linear in δ.

Theorem 8 (([28] Theorems 1 & 4) ESS have PC^r flowboxes). *Given $F : D \to \mathsf{T}D$ an ESS near ρ. (1) there exists a flow $\phi : \mathcal{F} \to D$ for F over a flow domain $\mathcal{F} \subset \mathbb{R} \times D$ containing $(0, \rho)$ such that $\phi \in PC^r(\mathcal{F}, D)$ and $\forall (t,x) \in \mathcal{F} : \phi(t,x) = x + \int_0^t F(\phi(s,x))\, ds$. (2) there exists a piecewise–differentiable homeomorphism $\psi \in PC^r(V,W)$ between neighborhoods $V \subset D$ of ρ and $W \subset \mathbb{R}^d$ of 0 such that $\psi \circ \phi(t,x) = \psi(x) + te_1$ where $e_1 \in \mathbb{R}^d$ is the first standard Euclidean basis vector.*

Orbital stability The stability of a limit cycle is a case of "*orbital stability*" – the stability of a trajectory under perturbations in initial state. For a smooth vector field $F : D \to \mathsf{T}D$, the flow is smoothly dependent on initial condition. The effect of initial perturbations to a trajectory $x : [0,T] \to D$ can be approximated to first order by solving the matrix valued "*variational equation*" (a linear time varying system) $\dot{S}(t) = \mathbf{D}F(x(t))S(t)$, giving the "*saltation matrix*" S. For an ESS adjustments must be made to the variational equation wherever a trajectory crosses a discontinuity in F; computational formulae for these are now known [[28] sec. 6.1 and 6.2]. Taking the simplest case – that of $m = 1$ – with s such that $\rho = x(s)$ and $h(\rho) = 0$, the adjustment would be: $S(s^+) = \left[I + \frac{(F(+1,\rho) - F(-1,\rho))\mathbf{D}h(\rho)}{\mathbf{D}h(\rho)F(-1,\rho)}\right] S(s^-)$.

4. THE TOOLBOX OF DATA DRIVEN FLOQUET ANALYSIS

DDFA comprises the application of a set of tools that produce progressively more detailed predictive models, using only observations of oscillator trajectories as input. At its most basic, the Floquet Normal Form describes states as perturbations away from the limit cycles. The initial step of DDFA is therefore to approximate the limit cycle, a process that requires filtering the noise out of measurements of limit cycle states. Presuming only stochastic measurement noise corrupts limit cycle data, this is equivalent to phase estimation: each measurement contributes to the estimate of some limit cycle points, and those in turn represent phase.

Given a phase estimate, one may anticipate future motions of the animal or robot by extrapolating the time evolution of phase and expecting the corresponding limit cycle state. This approach, and the use of phase estimates to predict animal motions has been described in detail[30] and used to analyze cockroach.[4, 31–34]

Following the construction of a limit cycle model, it is natural to inquire as to the dimension of the actual dynamics of the oscillator. The dimension of the slow dynamics can be identified through the statistical signature of return map Jacobian eigenvalues diverging from those of a random matrix.[4] Substantial problems remain in obtaining reliable estimates of return map linearizations. Several authors have noted that done naively, the observed eigenvalues change with phase[14, 35–37] – a result contrary to theory (see §2). Estimating the return map by using multiple Poincaré sections simultaneously[35, 37] improves the reliability of the results.

The estimation of a linearized return map from an isochron to itself is often no more difficult than estimation of affine maps between any two isochrons near the limit cycle. Such maps take the limit cycle points to each other and map the tangent spaces of one isochron into the other. The component of these maps acting transversely to the limit cycle is the system matrix of the LTP system of equation (1). A complete DDFA model consists of a phase estimator for $\mathcal{B}$, a limit cycle estimate $\hat{\gamma}(\cdot)$, a return map linearization estimate written in terms of its eigenvectors $\hat{Q}\hat{\Lambda}\hat{Q}^{-1}$, and linearized maps $\hat{\Phi}$ relating the isochrons to each other (i.e. $\hat{\Phi} : [0,T] \times [0,T] \to L(\mathcal{B},\mathcal{B})$ is such that $\hat{\Phi}(t,s)(x(s) - \gamma(s)) \approx (x(t) - \gamma(t))$ to first order, for any trajectory $x(\cdot)$).

DDFA models have been used to directly estimate the connection, by looking at the "*Impact of a Floquet Mode*"[14] – the change in $\mathcal{Q}$ associated with a unit perturbation along an eigendirection of the linearized return map (a column of $\hat{Q}$) inside of $\mathcal{B}$. Each impact is a column of the linear approximation to the connection at a given phase.

The DDFA model is, generally speaking, a good predictor of future oscillator states. It may be used as a "gold standard" for validating and improving the predictive capabilities of simpler models. We recently used DDFA to provide evidence in support of a 2009 prediction[38] that control of human running allows complete ("dead-beat") recovery within 2 steps.[37] We also used the DDFA to produce a linearized controller model for the human,[37] which could then be reduced into low dimensional "factors", allowing us to identify that by adding the state of the swing-leg ankle to the existing model's state-space the predictive ability of the model could be greatly improved.

5. SUMMARY

Floquet theory provides a coherent theoretical framework for data-driven modeling of legged locomotion, using the tools of Data Driven Floquet Analysis. As reviewed herein, the classical theory of nonlinear oscillators guarantees that exponentially stable oscillators admit full linearization by change of coordinates. Our results from recent work show that this statement also applies to the long-term dynamics of legged locomotion model, despite the of hybrid structure of the oscillators that represent legged systems.

ACKNOWLEDGMENTS

We wish to thank John Guckenheimer, Sam Burden, and Moritz Maus for their contributions to the work presented herein. This research was supported in part by ARO Young Investigator Award #61770 to S. Revzen, Army Research Laboratory Cooperative Agreements W911NF–08–2–0004 and W911NF–10–2–0016; and National Science Foundation Award #1028237. The views and conclusions contained in this document are those of the authors and should not be interpreted as representing the official policies, either expressed or implied, of the Army Research Laboratory or the U.S. Government. The U.S. Government is authorized to reproduce and distribute for Government purposes notwithstanding any copyright notation herein.

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